Generic fibers of the generalized Springer resolution of type A

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Abstract

It is well known that when the Lie algebra is of type $A$, $D$, $E$ the Springer fiber above a subregular nilpotent element is described by the Dynkin diagram and is called the Dynkin curve of the Lie algebra. On the other hand, the closure of the minimal nilpotent orbit is obtained by collapsing the zero section of a cotangent bundle of a projective space $\mathbb{P}^k$. In this article, we are interested in the study of the generalized Springer resolution of type $A$, we give a complete description of the generalized Springer fiber above a generic singularities showing that it is isomorphic to a Dynkin curve or to a projective space.

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1. Introduction and notations

In 1970, E. Brieskorn has discovered a connection between the rational double points singularities with the complex Lie algebra theory, (cf. [5]). His result is the following: let $G$ be a simple algebraic group of type $A$, $D$, $E$ with Lie algebra $\text{Lie}(G) = \mathfrak{g}$. Let $\mathcal{N}$ be the nilpotent cone of $\mathfrak{g}$. The variety $\mathcal{N}$ is exactly the closure of an unique nilpotent orbit $\mathcal{O}_{\text{reg}}$ called the regular nilpotent orbit. There is an unique nilpotent orbit $\mathcal{O}_{s-\text{reg}}$ of codimension 2 in $\mathcal{N}$ such that $\overline{\mathcal{O}_{s-\text{reg}}} = \mathcal{N} - \mathcal{O}_{\text{reg}}$ ($\mathcal{O}_{s-\text{reg}}$ is called the subregular nilpotent orbit). Let $T_x$ denote a transverse slice in $\mathfrak{g}$ to the orbit $\mathcal{O}_{s-\text{reg}}$ at the point

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\( x \in \mathcal{O}_{s_{\text{reg}}}. \) Then \( (T_x \cap \mathcal{N}, x) \) is a normal surface with an isolated rational double point of type corresponding to \( g. \) Few years later Esnault [10] has obtained the same result with a geometric point of view which consists to the study of the Springer resolution, \( f_b : T^*(G/B) \to \mathcal{N}, \) where \( B \) is a Borel subgroup of \( G \) [23]: the Springer fiber above \( x \in \mathcal{O}_{s_{\text{reg}}} \) is well known as a finite union of projective lines which corresponds to the Dynkin curve of \( g, \) and it was originally obtained by Tits (see [26]); H. Esnault shows that each projective line of the Springer fiber above a subregular nilpotent element has a self-intersection \(-2,\) this proves that the Springer resolution restricts to the minimal resolution of the generic singularities of \( \mathcal{N} \) and shows again that these singularities are rational double points of same type as \( g. \)

On the other hand, there is an other interesting singularity arising from the closure of the minimal nilpotent orbit \( \mathcal{O}_{\text{min}} \) in \( g \) corresponding to the unique (non-zero) nilpotent orbit which is contained in the closure of all non-zero nilpotent orbit, and \( \mathcal{O}_{\text{min}} = \mathcal{O}_{\text{min}} \cup \{0\} \) is normal and has an isolated singularity. In case \( g = \mathfrak{sl}(n, \mathbb{C}) \) such singularity is exactly obtained by collapsing the cotangent bundle of \( \mathbb{P}^{n-1}, \) so the fiber above such singularity is exactly the zero section of this cotangent bundle.

In the present work, we are interested in the study of the fibers of the generalized Springer resolution, \( f_p : T^*(G/P) \to \mathcal{O}_p, \) where \( P \) is a parabolic subgroup of a semisimple complex algebraic group \( G \) and \( \mathcal{O}_p \) denotes the Richardson orbit associated to \( \text{Lie}(P) = p. \) Firstly, we obtain a result on the dimension of the fibers of \( f_p, \) (cf. Theorem 2.1) which is a generalization of a Steinberg’s work [26,27], the last result will allow us to describe some irreducible components of the fibers of \( f_p \) (cf. Proposition 2.4). Next, we restrict our study to the case \( G = \text{SL}(n, \mathbb{C}); \) we will give a description of the intersection \( \mathcal{O}_p \) with the nilpotent radical of \( p \) (cf. Theorem 3.3), this will help us to describe the closure of the intersection of the nilpotent radical of \( p \) with every adjacent nilpotent orbit to \( \mathcal{O}_p \) (cf. Theorem 3.7), and we will give a complete description of the generalized Springer fibers above the elements of such orbit (cf. Theorem 3.9) showing that those fibers are isomorphic to a Dynkin curve or to a projective space. Finally, by adopting Esnault’s work we will find in some cases that the generalized Springer resolution restricts to the minimal resolution of some rational double points of type \( A \) (cf. Theorem 4.6).

Let \( G \) be a semisimple (connected) complex algebraic group with Lie algebra \( \text{Lie}(G) = \mathfrak{g} \) on which \( G \) acts by the adjoint action. Fix a Cartan subalgebra \( \mathfrak{h}. \) Let \( \mathcal{W} \) denote the associated Weyl group. We have the Chevalley–Cartan decomposition of \( \mathfrak{g}: \)

\[
\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha,
\]

where \( \mathcal{R} \) is the root system of \( \mathfrak{g} \) relatively to \( \mathfrak{h}. \) Let \( S \) be a set of simple roots of \( \mathcal{R}. \) Denote \( \mathcal{R}^+ \) (resp., \( \mathcal{R}^- \)) the positive roots (resp., negative roots) (w.r.t. \( S). \) Let \( b := \mathfrak{h} \oplus \sum_{\alpha \in \mathcal{R}^-} \mathfrak{g}_\alpha \) be the standard Borel subalgebra (w.r.t. \( S). \) Let \( B \) be the Borel subgroup of \( G \) with \( \text{Lie}(B) = b. \) Let \( P \) be a standard parabolic subgroup with \( \text{Lie}(P) = p. \) The parabolic subalgebra \( p \) is determined by a subset \( S_p \subset S. \) Denote \( \mathcal{R}_p \) the root subsystem generated by \( S_p \) and \( \mathcal{W}_p \) the subgroup of \( \mathcal{W} \) generated by the simple reflections \( s_\alpha \)
with \( \alpha \in R_p \). We also have

\[ p = l_p \oplus n_p \text{ with } l_p := h \oplus \sum_{\alpha \in R_p} g_\alpha \text{ and } n_p := \sum_{\alpha \in R^+ - R_p} g_\alpha, \]

where \( l_p \) is the Levi component and \( n_p \) is the nilpotent radical of \( p \). Denote \( L_p \) and \( U_p \), the connected algebraic subgroups of \( G \) with \( \text{Lie}(L_p) = l_p \) and \( \text{Lie}(U_p) = n_p \).

Denote \( \mathcal{W}^S_p \) (resp., \( S_p \mathcal{W}^S_p \)) the set of the representatives of minimal length of the classes of \( \mathcal{W} / \mathcal{W}_p \) (resp., of \( \mathcal{W}_p \setminus \mathcal{W} / \mathcal{W}_p \)). We have

\[ \mathcal{W}^S_p = \{ w \in \mathcal{W}; w(S_p) \subset R^+ \}, \quad (1.1) \]

\[ S_p \mathcal{W}^S_p = \{ w \in \mathcal{W}; w(S_p) \subset R^+ \text{ and } w^{-1}(S_p) \subset R^+ \}. \quad (1.2) \]

For every \( \alpha \in R \) denote \( U_\alpha \) the unique unipotent subgroup of \( G \) such that \( \text{Lie}(U_\alpha) = g_\alpha \). For every \( w \in \mathcal{W}^S_p \), let \( n_w \) denote a representative of \( w \) in \( \text{Norm}_G(h) \); define

\[ N_p(w) := \{ \alpha \in R^+ | w^{-1}(\alpha) \in R^- - R_p \} \quad (1.3) \]

and \( U_{p,w} \) the unipotent subgroup of \( G \) generated by the subgroups \( U_\alpha \) with \( \alpha \in N_p(w) \).

We have the well-known Bruhat–Tits decomposition (see [2, p.100]). Every element \( g \) in \( G \) can be uniquely written as the product \( g = unwp \), with \( w \in \mathcal{W}^S_p \), \( u \in U_{p,w} \), and \( p \in P \), and we also have

\[ G = \coprod_{w \in \mathcal{W}^S_p} Pn_wP. \quad (1.4) \]

From general nilpotent orbit theory, recall that there is a unique nilpotent \( G \)-orbit \( \mathcal{O}_p \) such that the set \( \mathcal{O}_p \cap n_p \) is open and dense in \( n_p \). Moreover, \( \mathcal{O}_p \cap n_p \) is exactly a \( P \)-orbit and we have \( \dim(\mathcal{O}_p) = 2\dim(n_p) \) (cf. [19,26]). \( \mathcal{O}_p \) is called the Richardson orbit associated to \( p \).

Let \( G \times^P n_p \) be the space obtained as the quotient of \( G \times n_p \) by the right action of \( P \) given by \( (g, x).p := (gp, p^{-1}.x) \) with \( g \in G, x \in n_p \) and \( p \in P \). By the Killing form we get the following identification \( G \times^P n_p \simeq T^*(G/P) \). Let \( g \ast x \) denote the class of \( (g, x) \) and \( \mathcal{P} := G / P. \) The map \( G \times^P n_p \to \mathcal{P} \times g, g \ast x \mapsto (gP, g.x) \) is an embedding which identify \( G \times^P n_p \) with the following closed subvariety of \( \mathcal{P} \times g \) (see [21, p. 19]):

\[ \mathcal{Y} := \{(gP, x) | x \in g.n_p\}. \quad (1.5) \]
The map $f_p : G \times \mathfrak{p} \to G$, $g \times x \mapsto g.x$ is called the generalized Springer resolution and we have the following commutative diagram:

$$
\begin{array}{ccc}
G \times \mathfrak{p} & \xrightarrow{\sim} & \mathcal{Y} \\
\downarrow f_p & & \downarrow \downarrow pr_2 \\
\mathfrak{g} & & \\
\end{array}
$$

(1.6)

where $pr_2$ is the second projection of $\mathcal{P} \times \mathfrak{g}$ on $\mathfrak{g}$. The map $f_p$ is proper (because $G/P$ is complete) and its image is exactly $G.\mathfrak{p} = \overline{O_p}$. Moreover, the fiber of $f_p$ above points of $\mathcal{O}_p$ is finite; it is a birational map when $G^x \subset P$, where $G^x$ is the stabilizer of $x$ in $G$ and $x \in \mathcal{O}_p$, this happens in the particular case $P = B$ is a Borel subgroup of $G$ [26, Theorem 1, p. 129], and in this case the map $f_b$ is a desingularization of the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$ and is called the Springer resolution [23].

In case $G = \text{SL}(n, \mathbb{C})$, the generalized Springer resolution is birational [3], moreover every nilpotent orbit is a Richardson orbit for an appropriate parabolic subgroup of $\text{SL}(n, \mathbb{C})$ [6, p. 112]. So the generalized Springer resolutions are the desingularizations of the closures of the nilpotent orbits.

Let $x$ be a nilpotent element in $\mathfrak{p}$. By (1.6) we have

$$
\mathcal{P}_x := f_p^{-1}(x) = \{ g.P \in \mathcal{P} \mid x \in g.\mathfrak{p} \},
$$

$$
\mathcal{P}_x = \{ g.P \in \mathcal{P} \mid g^{-1}.x \in \mathfrak{p} \}. 
$$

(1.7)

Following [22], let $G^x$ be the stabilizer of $x$ in $G$, $G^{x,0}$ denote its neutral component and $A(x) := G^x / G^{x,0}$ the component group. Let $\{C_\sigma\}_{\sigma \in H}$ (resp., $\{D_i\}_{1 \leq i \leq m}$) denote the set of the irreducible components of $\mathcal{P}_x$ (resp., of $G.x \cap \mathfrak{p}$). We have a surjective map $\pi : H \to \{1, \ldots, m\}$ such that for every $1 \leq i \leq m$ the set $H_i := \pi^{-1}(i)$ is exactly an orbit under the action of $A(x)$.

In case $G = \text{SL}(n, \mathbb{C})$, the subgroup $G^x \subset \text{GL}(n, \mathbb{C})^x$ is always connected (it is an open set of the space $\{g \in \text{gl}(n, \mathbb{C}) \mid gx = xg\}$), then $\pi$ is a bijection between the irreducible components of $\mathcal{P}_x$ and the irreducible components of $G.x \cap \mathfrak{p}$.

Let us give a brief outline of the contents of the paper.

• In Section 2, we will give a fundamental result about the dimension of the generalized Springer resolution; this will help us to give a description of some irreducible components of the generalized Springer fibers.
• In Section 3, we are interested in case $G = \text{SL}(n, \mathbb{C})$. Our main result gives a complete description of the generalized Springer fibers for elements in an adjacent nilpotent orbit to $\mathcal{O}_p$. 

• In Section 4, we adopt Esnault’s work to show in some cases that the generalized Springer resolution restricts to the minimal resolution of some rational double points of type $A$.

2. Generalities

The Springer resolution has been intensively studied by many mathematicians as N. Spaltenstein, G. Kempf, R. Steinberg, P. Slodowy,.... R. Steinberg has established the following formula which related the dimension of the Springer fiber above a nilpotent element with the dimension of the stabilizer of $x$ in $G$: $\dim(f_b^{-1}(x)) = \frac{1}{2}(\dim(G^x) - r)$, where $r$ is the rank of $G$ (cf. [27, p. 133; 27, p. 217]). By studying his proof we have obtained the following generalization:

**Theorem 2.1.** For every element $x \in \overline{O}_p = \text{Im} \ f_p$ we have

$$\dim(f_p^{-1}(x)) \leq \frac{1}{2}(\dim(G^x) - \dim(l_p)).$$

**Proof.** For every $x \in \overline{O}_p$, denote $O_x$ the $G$-orbit of $x$. Consider the subvariety $V$ of $g \times P \times P$ defined by

$$V := \{(y, g.P, g'.P) \in O_x \times P \times P; \ y \in g.n_p \cap g'.n_p \}.$$  

Then $V$ is a closed $G$-variety and is a fibration above $O_x$ whose fibers are isomorphic to $P_x \times P_x$. We deduce that

$$\dim(V) = 2\dim(P_x) + \dim(O_x). \quad (2.1)$$

By the Bruhat–Tits decomposition we have a disjoint union $V = \bigsqcup_{w \in S_P \mathcal{W}_P} V_w$. Let $w \in S_P \mathcal{W}_P$ and let $n_w$ be a representative of $w$ in $\text{Norm}_G(h)$, then $V_w := \{(y, g.P, g.n_w.P) \in V \}$. In particular, we have

$$\dim(V) = \max_{w \in S_P \mathcal{W}_P} \dim(V_w).$$

We can identify $V_w$ with a subvariety of $O_x \times [G/(P \cap n_w.P)]$ by the following morphism:

$$\phi : V_w \to O_x \times [G/(P \cap n_w.P)],$$

$$(y, g.P, g.n_w.P) \mapsto (y, g(P \cap n_w.P)).$$

Moreover, the projection $p : V_w \to G/(P \cap n_w.P)$ allows us to see that the fiber above $g(P \cap n_w.P)$ is exactly $O_x \cap g.n_p \cap g.n_w.n_p \simeq O_x \cap n_p \cap n_w.n_p$. Then $V_w$ is a $G$-bundle.
above the space $G/(P \cap n_w.P)$ with the fibers isomorphic to $O_x \cap n_p \cap n_w.n_p$. We deduce that

$$\dim(V_w) = \dim(G) - \dim(P \cap n_w.P) + \dim(O_x \cap n_p \cap n_w.n_p).$$  \hfill (2.2)

But

$$\dim(P \cap n_w.P) = \dim(p \cap n_w.p),$$

$$\dim(p \cap n_w.p) = \dim(p \cap n_w.I_p) + \dim(I_p \cap n_w.n_p) + \dim(n_p \cap n_w.n_p).$$

The element $w \in W$ permutes the roots, so $n_w.I_p = h \oplus \sum_{x \in R_p} g_{w(x)}$ and the lines $g_w(x)$ which are not contained in $p$ are exactly those for which $w(x) \in R^- - R_p^-$. We deduce that

$$\dim(p \cap n_w.I_p) = \dim(I_p) - \text{card}\{x \in R_p; \ w(x) \in R^- - R_p^- \}.$$  \hfill (2.3)

As $w \in S_p W S_p$ (cf. 1.2) we get $\dim(p \cap n_w.I_p) = \dim(I_p) - \text{card}\{x \in R_p^+; \ w^{-1}(x) \in R^+ - R_p^+ \}$.  \hfill (2.5)
and
\[ \dim(I_p \cap n_{w-1}.n_p) = \text{card}\{ \beta \in \mathcal{R}_p^+; \ w(\beta) \in \mathcal{R}^+ - \mathcal{R}_{-p}^+ \}. \]
(2.6)

On the other hand,
\[ 2\dim(p \cap n_w.p) = \dim(p \cap n_w.p) + \dim(p \cap n_{w-1}.p). \]

With (2.3), (2.4), (2.5) and (2.6) we get
\[ 2\dim(p \cap n_w.p) = 2\dim(I_p) + 2\dim(n_p \cap n_w.n_p). \]
(2.7)

With the relations (2.2) and (2.3) we get
\[ \dim(V_w) = \dim(G) - \dim(I_p) - \dim(n_p \cap n_w.n_p) + \dim(O_x \cap n_p \cap n_w.n_p). \]

As \( \dim(n_p \cap n_w.n_p) - \dim(O_x \cap n_p \cap n_w.n_p) \geq 0 \), with (2.1) and \( \dim(O_x) = \dim(G) - \dim(G^x) \) we deduce that
\[ \dim(\mathcal{P}_x) \leq \frac{1}{2}(\dim(G^x) - \dim(I_p)). \]
\[ \square \]

**Remark 2.2.** The author thanks the anonymous referee for indicating that the last theorem was a special case of a result obtained by Springer (see [24, Lemma 4.2; 18, Proposition 1.2]).

Moreover, the above relation is an equality if \( \dim(n_p \cap n_w.n_p) - \dim(O_x \cap n_p \cap n_w.n_p) = 0 \), so we have

**Corollary 2.3.** For every element \( x \in \overline{O}_p \) we have: \( \dim(f_p^{-1}(x)) = \frac{1}{2}(\dim(G^x) - \dim(I_p)) \) if and only if \( O_x \cap n_p \cap n_w.n_p \) is dense in \( n_p \cap n_w.n_p \) for an element \( w \in S_p \).

An immediate application of this theorem is the possibility to describe certain irreducible components of the fibers of the generalized Springer resolution \( f_p \).

**Proposition 2.4.** Let \( P \) be a standard parabolic subgroup. Let \( Q \) be a parabolic subgroup which contains \( P \) and let \( Q' \) be a parabolic subgroup in the conjugacy class of \( Q \). Denote \( \text{Lie}(P) = p, \text{Lie}(Q) = q \) and \( \text{Lie}(Q') = q' \). Let \( n_{q'} \) be the nilpotent radical of \( q' \). Let \( O_q \) be the Richardson orbit associated to \( q \). Let \( x \) be a nilpotent element. Then we have the following equivalences:

(i) \( x \in n_{q'} \cap \overline{O}_q \).
(ii) \( x \in n_{q'} \) and \( \dim(Q/P) = \dim(f_p^{-1}(x)) \).
(iii) \( g.Q/P \) is an irreducible component of \( f_p^{-1}(x) \) where \( g \) is an element in \( G \) such that \( gQg^{-1} = Q' \).

**Proof.** (i) \(\Rightarrow\) (ii) Let \( S \) be the set of simple roots of \( \mathcal{R} \). Denote \( S_q := \{ \alpha \in S \mid \alpha \in \mathcal{R}_q \} \). Then \( S_q \) is a basis of the root subsystem \( \mathcal{R}_q \). Relatively to \( S \) (resp., \( S_q \)), denote \( l \) (resp., \( l_q \)) the length function on \( \mathcal{W} \) (resp., \( \mathcal{W}_q \)). Denote \( n^+ := \sum_{\alpha \in \mathcal{R}_q^+} g_\alpha \). Let \( w_q \) the unique element in \( \mathcal{W}_q \) such that \( w_q(\mathcal{R}_q^+) = \mathcal{R}_q^- \). As \( w_q \in \mathcal{W}_q \), then we have \([15, p. 114]\)

\[
l_q(w_q) = \text{card}(\{ \alpha \in \mathcal{R}_q^+ \mid w_q^{-1}(\alpha) \in \mathcal{R}_q^- \}) = \text{card}(\mathcal{R}_q^+).
\]

But \( l_q \) is only the restriction of \( l \) on \( \mathcal{W}_q \), \([15, p. 19]\), then we deduce that

\[
l_q(w_q) = l(w_q) = \text{card}(\{ \alpha \in \mathcal{R}_q^+ \mid w_q^{-1}(\alpha) \in \mathcal{R}_q^- \}).
\]

As consequence we have \( \mathcal{R}_q^+ \cap w_q(\mathcal{R}_q^+) = \mathcal{R}_q^+ - \mathcal{R}_q^- \). Denote \( w_q \) the unique element in \( \mathcal{W} \) of minimal length in the double class of \( w_q \) in \( \mathcal{W}_p \mathcal{W}/\mathcal{W}_p \). Then we have \( w_q = w_1w_qw_2 \) with \( w_1, w_2 \in \mathcal{W}_p \). We deduce that for every \( \alpha \in \mathcal{R}_q^+ \) we have \( w_qw_2(\alpha) \in \mathcal{R}_p \). By the same argument we have \( w_2(\mathcal{R}_q^+ - \mathcal{R}_p^+) = \mathcal{R}_q^- - \mathcal{R}_p^- \), we deduce that \( w_q(\mathcal{R}_q^+ - \mathcal{R}_p^+) \subset \mathcal{R}_p \). As consequence we have

\[
n_p \cap w_q(n_p) = n_q. \tag{2.8}
\]

By Corollary 2.3 we have

\[
\dim(f_p^{-1}(x)) = \frac{1}{2}(\dim(G^+) - \dim(l_p)).
\]

With properties on Richardson orbit we can verify that \( \dim(G^+) = \dim(l_q) \).

\[
\dim(f_p^{-1}(x)) = \frac{1}{2}(\dim(l_q) - \dim(l_p)) = \dim(Q/P).
\]

(ii) \(\Rightarrow\) (iii) is trivial.

(iii) \(\Rightarrow\) (i) Say that \( g.Q/P \) is an irreducible component of \( f_p^{-1}(x) \) is equivalent to say that \( Q/P \) is an irreducible component of \( f_p(g^{-1}.x) \), by (1.7) we have \( g^{-1}.x \in n_p \). By (2.8) we can conjugate \( g^{-1}.x \) with an element of \( Q \) to assume that we have \( g^{-1}.x \in n_q \) (if not by (1.7) we would have \( f_p^{-1}(x) = \emptyset \)), and we have

\[
\dim(Q/P) = \dim(f_p^{-1}(x)).
\]

Then we have

\[
\dim(l_p) + 2\dim(f_p^{-1}(x)) = \dim(G^+) \geq \dim(Pg^{-1}.x).
\]
As consequence
\[
\dim(Q.(g^{-1}.x)) = \dim(Q) - \dim(Q^{g^{-1}.x}) \geq \dim(Q) - (\dim(l_p) + 2\dim(f_p^{-1}(x))) = \dim(n_q).
\]

Now by properties on Richardson orbit we get the result. \( \square \)

**Remark 2.5.** Let \( x \) be an element of \( g \). A *polarization* of \( x \), is a Lie subalgebra \( q \) of \( g \) such that \( \kappa(x, [q, q]) = 0 \) and \( 2\dim(q) = \dim(q^x) + \dim(g) \) where \( \kappa(\, , \,) \) is the Killing form. Then every polarization is necessary a parabolic subalgebra and the nilpotent elements which admit polarizations are exactly the nilpotent elements of Richardson orbits, [7, p. 46]. Then the above proposition says that the different polarizations of \( x \) which contain \( p \), give certain irreducible components of \( f_p^{-1}(x) \).

### 3. Study in \( \mathfrak{s}l(n, \mathbb{C}) \)

Now consider the case \( G = SL(n, \mathbb{C}) \) and \( \mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}) \). The subalgebra \( \mathfrak{h} \) (resp., \( \mathfrak{b} \)) can be identified with the subvariety which consists of the diagonal matrices (resp., upper triangular matrices) of \( \mathfrak{sl}(n, \mathbb{C}) \). Denote \( E_{i,j} \) the elementary matrices. The one-dimensional vector subspaces \( \mathfrak{g}_2 \) are generated by the elementary matrices \( E_{i,j} \) with \( i \neq j \). For every \( 1 \leq i, j \leq n \), denote \( p_{i,j} \) the coordinate projection corresponding to the line \( \mathfrak{g}_2 \) generated by the elementary matrix \( E_{i,j} \). The roots are given by the following linear forms \( \{ p_{i,i} - p_{j,j} \} \), with \( i \neq j \). The simple roots \( \{ \varepsilon_i \}_{i=1,\ldots,n-1} \) are the linear form \( \{ p_{i,i} - p_{i+1,i+1} \}_{i=1,\ldots,n-1} \), and the Weyl group is identified with the symmetric group \( \mathfrak{S}_n \), [4, p. 250/251]. Let \( s_k \) be the elementary transposition of \( \mathfrak{S}_n \) which interchanges \( k \) and \( k + 1 \).

The reasons to consider \( \mathfrak{sl}(n, \mathbb{C}) \) are on the one hand the generalized Springer resolution is a desingularization, and on the other hand every nilpotent orbit is a Richardson orbit for a suitable parabolic subalgebra, [6, p. 112].

**Definition 3.1.** A partition of \( n \) is a sequence of integers \( p = (p_1, p_2, \ldots, p_l) \) such that \( p_i \geq 1 \) and \( \sum_{i=1}^l p_i = n \).

The standard parabolic subalgebras of \( \mathfrak{sl}(n, \mathbb{C}) \) are in bijective correspondence with the partitions of \( n \). If \( p = (p_1, p_2, \ldots, p_l) \) is a partition of \( n \), then the corresponding standard parabolic subalgebra has the following shape:

\[
\begin{pmatrix}
L_1 & * & \ldots & * \\
0 & L_2 & * & \\
\vdots & \vdots & \ddots & * \\
0 & \ldots & 0 & L_l
\end{pmatrix}
\]
where $L_i \in \text{GL}_{p_i \times p_i}(\mathbb{C})$. Two partitions $p = (p_1, p_2, \ldots, p_l)$ and $q = (q_1, q_2, \ldots, q_l)$ of $n$ are called associated if there is permutation $\sigma \in \mathfrak{S}_l$ such that $q_i = p_{\sigma(i)}$. A partition $p = (p_1, p_2, \ldots, p_l)$ of $n$ is said ordered if $p_1 \geq p_2 \geq \cdots \geq p_l$. To the partition $p$ corresponds the Young diagram whose rows are composed respectively, of $p_1, p_2, \ldots, p_l$ squares. If $p = (p_1, p_2, \ldots, p_l)$ is an ordered partition of $n$ we define its dual partition as the partition $\hat{p} = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_l)$ with $\hat{p}_i := \text{card}\{j; \ p_j \geq i\}$. We can notice that the dual partition is also ordered.

The nilpotent orbits in $\mathfrak{sl}(n, \mathbb{C})$ are parameterized by the ordered partitions of $n$ [6, p. 32] corresponding to the lengths of the Jordan blocs arranged in decreasing order; if $p$ is an ordered partition of $n$, we denote $O_p$ the corresponding nilpotent orbit. We have the following identities (see [6, p. 94]):

$$\dim(\ker(x^j)) = \sum_{i=1}^{j} \hat{p}_i$$

and

$$\text{rank}(x^j) = \sum_{i>j} \hat{p}_i. \quad (3.1)$$

If $p = (p_1, p_2, \ldots, p_l)$ and $q = (q_1, q_2, \ldots, q_k)$ are two ordered partition of $n$, we denote $p \succeq q$ if

$$\sum_{i=1}^{j} p_i \geq \sum_{i=1}^{j} q_i \quad \text{for every} \quad j. \quad (3.2)$$

If $p \succeq q$, we will say that the partition $p$ dominates the partition $q$.

The geometric interpretation of this order is given by

**Proposition 3.2** (Gerstenhaber [6, p. 95; 11]).

(i) $p \succeq q$ if and only if $O_p \supset O_q$.

(ii) If $p \succeq q$ such that for every $O_q \subset O \subset O_p$ we have $O = O_q$ or $O = O_p$. Then we have:

Case: 1 there is an integer $i$ such that $p_k = q_k$ for $k \neq i, i + 1$ and $q_i = p_i - 1 \geq q_{i+1} = p_{i+1} + 1$. Then we have $\text{codim}_{O_p}(O_q) = 2$.

Case: 2 there is two integers $i < j$ such that $p_k = q_k$ for $k \neq i, j$ and $q_i = p_i - 1 = q_j = p_j + 1$. Then we have $\text{codim}_{O_p}(O_q) = 2(j - i)$.

Such partitions are called “adjacent”.
We can see the two cases by their Young diagrams in the following manner:

**Case: 1**

\[
\begin{array}{c}
\text{p} = \\
\end{array}
\begin{array}{c}
\text{q} = \\
\end{array}
\]

**Case: 2**

\[
\begin{array}{c}
\text{p} = \\
\end{array}
\begin{array}{c}
\text{q} = \\
\end{array}
\]

The first case consists to move a box in a corner to the next row, and the second case consists to move a box in a corner to the previous column.

If \( p \) is an ordered partition of \( n \), then the nilpotent orbit \( O_p \) is the Richardson orbit for every standard parabolic subalgebra whose corresponding partition is associated to \( \hat{p} \) (cf. [6, p. 112]), in particular we have \( O_p = O_\hat{p} \).

Let \( p \) be an ordered partition of \( n \) and let \( p \) be the standard parabolic subalgebra corresponding to the partition \( \hat{p} \). We have the decomposition \( p = I_p \oplus n_p \). As the subalgebra \([n_p, n_p]\) is stable under \( L_p \) which is a reductive group, there is a vector subspace \( V_p \) such that \( n_p = V_p \oplus [n_p, n_p] \) and \( V_p \) is stable under \( L_p \). In fact \( V_p \) is unique, it is the direct sum of the subspaces \( \alpha_x \), where \( x \) is the sum of simple roots in \( S_p \) and of a unique simple root in \( S - S_p \), [6, p. 123]. Here is the first important result:

**Theorem 3.3.** Let \( p = (p_1, p_2, \ldots, p_l) \) an ordered partition of \( n \). Let \( O_p \) be the nilpotent orbit corresponding to \( p \). Let \( p \) be the parabolic subalgebra corresponding to \( \hat{p} = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_l) \). Then we have:

(i) The subvariety \( O_p \cap V_p \) is reduced to a unique \( L_p \)-orbit which is open and dense in \( V_p \).

(ii) \( O_p \cap n_p = (O_p \cap V_p) \oplus [n_p, n_p] := \{x + y; \ x \in O_p \cap V_p, \ y \in [n_p, n_p]\} \).

**Proof.** (i) Let \( x = x_1 + x_2 \in n_p = V_p \oplus [n_p, n_p] \) be an element in \( n_p \) with \( x_1 \in V_p \) and \( x_2 \in [n_p, n_p] \), then \( x \in O_p \) if and only if \([p, x] = n_p\). Now \( V_p \) is stable under
we have $[I_p, x_1] \subset V_p$, the condition

$$[p, x] = [I_p, x_1] + [I_p, x_2] + [n_p, x_1] + [n_p, x_2] = n_p$$

with

$$[I_p, x_1] \subset V_p, \quad [I_p, x_2] + [n_p, x_1] + [n_p, x_2] \subset [n_p, n_p]$$

implies $[I_p, x_1] = V_p$, but this last equality is equivalent to the fact that the orbit $O_2$ of $x_1$ under $L_p$ is open and dense in $V_p$. So if we denote $p : n_p = V_p \oplus [n_p, n_p] \to V_p, x_1 + x_2 \mapsto x_1$ the first projection, we get $p(O_p \cap n_p) \subset O_2$, in particular we deduce that $L_p, p(O_p \cap n_p) = O_2$. We can easily verify that the elements $x_1 \in V_p$ have the following shape:

$$\begin{pmatrix}
0 & M_1^1 & 0 & \ldots & 0 \\
0 & 0 & M_2^1 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
\vdots & \vdots & \ddots & 0 & M_{t-1}^1 \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix}$$

(3.3)

with $M_j^1 \in \text{Mat}_{\hat{p}_j \times \hat{p}_{j+1}}(\mathbb{C})$ for $1 \leq j \leq t-1$, we can identify $V_p \simeq \text{Mat}_{\hat{p}_1 \times \hat{p}_2}(\mathbb{C}) \times \cdots \times \text{Mat}_{\hat{p}_{t-1} \times \hat{p}_t}(\mathbb{C})$ and write

$$x_1 = (M_1^1, \ldots, M_{t-1}^1) \in \text{Mat}_{\hat{p}_1 \times \hat{p}_2}(\mathbb{C}) \times \cdots \times \text{Mat}_{\hat{p}_{t-1} \times \hat{p}_t}(\mathbb{C}).$$

(3.4)

Now it is easy to see that if we consider $x_1 \in V_p$ with the configuration (3.3), then we have

$$x_1^2 = \begin{pmatrix}
0 & 0 & M_1^2 & 0 & \ldots & 0 \\
0 & 0 & 0 & M_2^2 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & 0 & M_{t-2}^2 \\
\vdots & \vdots & \ddots & \ddots & 0 & 0 \\
0 & \ldots & \ldots & \ldots & 0 & 0
\end{pmatrix}$$

(3.5)
with $M^i_t := M^i_1 M^i_{t+1} \in \text{Mat}_{\hat{p}_i \times \hat{p}_{i+2}}(\mathbb{C})$, so we can write $x^2 = (M^1_i, \ldots, M^2_{t-2}) \in \text{Mat}_{\hat{p}_1 \times \hat{p}_2}(\mathbb{C}) \times \cdots \times \text{Mat}_{\hat{p}_{t-2} \times \hat{p}_t}(\mathbb{C})$. Then by induction we can verify that

$$x^k_1 = (M^k_1, \ldots, M^k_{t-k}) \in \text{Mat}_{\hat{p}_1 \times \hat{p}_{k+1}}(\mathbb{C}) \times \cdots \times \text{Mat}_{\hat{p}_{t-k} \times \hat{p}_t}(\mathbb{C})$$  \hspace{1cm} (3.6)

with $M^k_i := M^1_1 M^1_{i+1} \cdots M^1_{i+k-1}$. Suppose that $x_1$ is of maximal rank. As $\text{rank}(x_1) = \sum_{1 \leq i \leq t} \text{rank}(M^1_i)$ and as $M^i_1 \in \text{Mat}_{\hat{p}_i \times \hat{p}_{i+1}}(\mathbb{C})$ with $\hat{p}_i \geq \hat{p}_{i+1}$, we deduce that $\text{rank}(x_1) = \sum_{i \geq 2} \hat{p}_i$. Likewise for every integer $k$, we have $\text{rank}(x^k_1) = \sum_{1 \leq i \leq t-k+1} \text{rank}(M^k_i)$, and as $M^k_i \in \text{Mat}_{\hat{p}_i \times \hat{p}_{i+k}}(\mathbb{C})$ with $\hat{p}_i \geq \hat{p}_{i+k}$ we get $\text{rank}(M^k_i) = \hat{p}_{i+k}$, so we get $\text{rank}(x^k_1) = \sum_{i \geq k+1} \hat{p}_i$. By (3.1) and Proposition 3.2, we deduce that $x_1 \in \mathcal{O}_2$ if and only if $x_1 \in \mathcal{O}_p$ if and only if $p(\mathcal{O}_p \cap \mathfrak{n}_p) = \mathcal{O}_2 = \mathcal{O}_p \cap V_p$ and this shows (i).

(ii) If we write $x = x_1 + x_2$ with $x_1 \in V_p$ and $x_2 \in \{\mathfrak{n}_p, \mathfrak{n}_q\}$, by the proof of (i) we have $x \in \mathcal{O}_p$ if and only if $x_1 \in \mathcal{O}_p$, and the result follows. \(\square\)

**Remark 3.4.** (i) The above theorem gives a characterization of the elements of the Richardson orbit $\mathcal{O}_p$; this will allow us to give a characterization of $\mathfrak{n}_p - (\mathcal{O}_p \cap \mathfrak{n}_p)$, in particular this will help us to find out the irreducible components of $G.\mathfrak{x} \cap \mathfrak{n}_p$ when $\mathfrak{x}$ is in an adjacent orbit to $\mathcal{O}_p$ (cf. Theorem 3.7).

(ii) We can also notice that this result is not always true for another parabolic subalgebra corresponding to a partition associated to $\hat{p}$. In the proof we use the fact that $\hat{p}$ is ordered, this permits us to show that $\mathcal{O}_p \cap V_p \neq \emptyset$.

Let $M$ be an irreducible subvariety contained in the nilpotent cone $\mathcal{N}$ of $\mathfrak{sl}(n, \mathbb{C})$. As $\mathcal{N}$ is a finite union of nilpotent orbits, there is a unique nilpotent orbit $\mathcal{O}_M$ such that $\mathcal{O}_M \cap M$ is dense in $M$.

**Definition 3.5.** We will call $\mathcal{O}_M$ the orbit induced by $M$.

Now let $p \geq q$ be two adjacent ordered partitions of $n$. Let $\mathfrak{p}$ be the standard parabolic subalgebra corresponding to the partition $\hat{p}$. As the image of the generalized Springer resolution $f_p$ is exactly $\overline{\mathcal{O}_p} = G.\mathfrak{n}_p$ (= $\mathcal{O}_p$), we deduce that $\mathcal{O}_q$ has a non-empty intersection with $\mathfrak{n}_p$. On the other hand as $q$ is adjacent to $p$, the orbit $\mathcal{O}_q$ is necessarily induced by every irreducible component of $\mathfrak{n}_p - (\mathcal{O}_p \cap \mathfrak{n}_p)$ for which the intersection with $\mathcal{O}_q$ is non-empty.

Denote $M(m, n) := \text{Mat}_{m \times n}(\mathbb{C})$. For every $0 \leq l \leq \min(m, n)$, denote $M_l(m, n) := \{x \in M(m, n) \mid \text{rank}(x) \leq l\}$; the subvariety $M_l(m, n)$ is called a determinantal variety, it is an irreducible normal subvariety of codimension $(m-l)(n-l)$ in $M(m, n)$, moreover $M_l(m, n)$ coincides with the closure of the subvariety of $M(m, n)$ which consists of matrices of rank equal to $l$ [1, Chapter II].

For every $1 \leq k \leq t-1$ let $Y_k$ denote the subvariety of $\mathfrak{n}_p$ defined by

$$Y_k := \{x_1 + x_2 \in V_p + [\mathfrak{n}_p, \mathfrak{n}_p] : \text{rank}(M^1_k) < \hat{p}_{k+1}\}.$$  \hspace{1cm} (3.7)
We can notice that $Y_k$ is exactly the direct sum of a determinantal variety and a vector space, therefore it is an irreducible normal subvariety of codimension $\hat{p}_k - \hat{p}_{k+1} + 1$ in $\mathfrak{n}_p$.

**Remark 3.6.** (i) By Theorem 3.3, the subvarieties $Y_k$ are exactly the irreducible components of $\mathfrak{n}_p - (\mathcal{O}_p \cap \mathfrak{n}_p)$. In case $\hat{p}_k = 1$, $Y_k$ is a hyperplane in $\mathfrak{n}_p$ and coincides with the nilpotent radical of a parabolic subalgebra of $\mathfrak{sl}(n, \mathbb{C})$ containing $p$.

(ii) Let $p \geq q$ be two-adjacent ordered partitions of $n$. If $\{Y_k\}_{k \in I}$ denotes the set of the irreducible components of $\mathfrak{n}_p - (\mathcal{O}_p \cap \mathfrak{n}_p)$ which induce $\mathcal{O}_q$, then we have $(\mathcal{O}_q \cap \mathfrak{n}_p) \subset \bigcup_{k \in I} Y_k$, in particular every irreducible component of $\mathcal{O}_q \cap \mathfrak{n}_p$ is (at least) contained in a subvariety $Y_k$ for a certain $k \in I$, and for every $k \in I$ the subvariety $Y_k$ contains a unique irreducible component of $\mathcal{O}_q \cap \mathfrak{n}_p$ which is dense in $Y_k$. As consequence, we have an injection from the set $I$ to the set of the irreducible components of $\mathcal{O}_q \cap \mathfrak{n}_p$. In particular, there is an injection from the set $I$ to the set of the irreducible components of $f_{\mathfrak{p}}^{-1}(x)$ for $x \in \mathcal{O}_q$. In fact, we will see at the end of the proof of Theorem 3.9 that we have in fact a bijection between these two sets.

**Theorem 3.7.** With the notations above. Denote $i_0 := \min\{j; \hat{p}_j \neq \hat{q}_j\}$ and $m_0 := \min\{j > i_0; \hat{p}_j \neq \hat{q}_j\}$. Then the irreducible components of $\mathfrak{n}_p - (\mathcal{O}_p \cap \mathfrak{n}_p)$ which induce $\mathcal{O}_q$ are all isomorphic and are the subvarieties $\{Y_k\}_{i_0 \leq k \leq m_0-1}$. Moreover we have $\dim(\mathcal{O}_q \cap \mathfrak{n}_p) = \frac{1}{2} \dim(\mathcal{O}_q)$.

**Proof.** We will consider two cases:

(i) Case $p_1 > q_1$: let $x = x_1 + x_2$ a nilpotent element in $\mathfrak{n}_p$, where $x_1 \in V_p$ with the formula

$$x_1 = (M^1_1, \ldots, M^1_{t-1}) \in \text{Mat}_{\hat{p}_1 \times \hat{p}_2}(\mathbb{C}) \times \cdots \times \text{Mat}_{\hat{p}_{t-1} \times \hat{p}_t}(\mathbb{C}).$$

(cf. (3.4)) and $x_2 \in [\mathfrak{n}_p, \mathfrak{n}_p]$. By Remark 3.6 (i), $x \in \mathfrak{n}_p - (\mathcal{O}_p \cap \mathfrak{n}_p)$ if and only if there is an integer $j \in \{1, \ldots, t - 1\}$ such that rank$(M^1_j) < \hat{p}_{j+1}$. By hypothesis $p_1 > q_1$, and by Proposition 3.2 we get

$$q_k = \begin{cases} p_1 - 1 & \text{if } k = 1, \\ p_2 + 1 & \text{if } k = 2, \\ p_k & \text{otherwise.} \end{cases} \quad (3.8)$$

With the dual partition we get $\hat{q}_{i_0} = 2$, $\hat{p}_{i_0} = 1$, $\hat{q}_k = \hat{p}_k$ for every $i_0 < k < t$ and $\hat{q}_t = 0$, $\hat{p}_t = 1$, in particular we deduce that for every $i_0 \leq k \leq t - 1$ we have $M^1_k \in \mathbb{C}$. So for every $i_0 \leq k \leq t - 1$, $Y_k = \{p_{i_0, \ldots, i_{k+1}} = 0\} \cap \mathfrak{n}_p$ is a hyperplane in $\mathfrak{n}_p$ with $l_k := \sum_{j \leq k} \hat{p}_j$ (cf. Remark 3.6 (i)), so $Y_k$ is exactly the nilpotent radical of the standard parabolic subalgebra corresponding to the set of simple roots $S_p \cup \{x_{l_k}\}$; now we can verify that such standard parabolic subalgebras correspond to partitions associated to $\hat{q}$. So these subvarieties $\{Y_k\}_{i_0 \leq k \leq t-1}$ induce the nilpotent orbit $\mathcal{O}_q$. Now, we have to show that the other irreducible components of $\mathfrak{n}_p - (\mathcal{O}_p \cap \mathfrak{n}_p)$ do not
induce $O_q$. Consider $k < i_0$ such that $\text{rank}(M^k) < \hat{p}_{k+1}$, by (3.6) we necessary have $\text{rank}(M^k) = \text{rank}(M^1 M^2 \ldots M^k) < \hat{p}_{j+1}$. In particular we have $\text{rank}(x^j) < \sum_{i > j} \hat{p}_i$. By (3.1), $x \in O_q$ if and only if $\text{rank}(x^j) = \sum_{i > j} \hat{q}_i$. But $j < i_0$, then $\hat{p}_i = \hat{q}_i$ for every $i \leq j < i_0$, as consequence $\sum_{i > j} \hat{q}_i = \sum_{i > j} \hat{p}_i$ and the result follows.

(ii) Case $p_1 = q_1$: we have $\hat{q}_{i_0} - 1 = \hat{p}_{i_0} = \hat{p}_{i_0+1} = \cdots = \hat{p}_{m_0-1} \geq 2$ and $\hat{q}_{m_0} = \hat{p}_{m_0} - 1$. Denote $X_k := \{ x \in \pi_\pi \mid \text{rank}(M^1) = \hat{p}_{k+1} - 1 \}$. Then by [1, p. 71], we have

$$Y_k = \overline{X_k}. \quad (3.9)$$

But for every $i_0 \leq k \leq m_0 - 1$, we can verify that $X_k \cap \pi_{\pi} \neq \emptyset$, where $\hat{p}$ is the standard parabolic subalgebra corresponding to the partition $(\hat{p}_1, \ldots, \hat{p}_{k-1}, \hat{p}_k + 1, \hat{p}_{k+1} - 1, \hat{p}_{k+2}, \ldots, \hat{p}_l)$, and the last partition is associated to the partition $\hat{q}$. By this remark and by (3.9) we deduce that $Y_k$ induces a nilpotent orbit $O \subset O_q$. Let $\pi_{\pi}$ be the nilpotent radical of the standard Borel subalgebra $b$. Then we have $\text{dim}(\pi_{\pi} \cap O) = \frac{1}{2} \text{dim}(O)$ [22]. As $Y_k \subset \pi_\pi \subset \pi_{\pi}$, we deduce that

$$\text{dim}(Y_k) \leq \frac{1}{2} \text{dim}(O) \leq \frac{1}{2} \text{dim}(O_q) = \text{dim}(\pi_{\pi})$$

the last equality comes from the properties of Richardson orbits. But we have noticed that the partition $(\hat{p}_1, \ldots, \hat{p}_{k-1}, \hat{p}_k + 1, \hat{p}_{k+1} - 1, \hat{p}_{k+2}, \ldots, \hat{p}_l)$ is associated to $\hat{q}$, we deduce that

$$\text{dim}(\pi_{\pi}) = \text{dim}(\pi_{\pi}) - \hat{p}_k + \hat{p}_{k+1} + 1.$$ 

As the variety $Y_k$ is of codimension $\hat{p}_k - \hat{p}_{k+1} + 1$, (cf. p. 450), then we have

$$\text{dim}(Y_k) = \frac{1}{2} \text{dim}(O) = \frac{1}{2} \text{dim}(O_q)$$

we deduce that $O = O_q$.

To finish the proof we have to verify that the other irreducible components $Y_j$ do not induce $O_q$ for $j \leq i_0 - 1$ or $m_0 \leq j$. But it is exactly the same reasoning as for the case (i) which consists to verify that $\text{dim}(x^j) < \sum_{l > j} \hat{q}_i$ for every element $x \in Y_j$. 

Finally, if $m_0 = i_0 + 1$ there is a unique irreducible component of $\pi_\pi - (O_\pi \cap \pi_\pi)$ which induces $O_q$, and if $m_0 \geq i_0 + 2$ for every $i_0 \leq i, j \leq m_0$ we have $\tilde{p}_i = \tilde{p}_j$ and as consequence the matrices $\{M^i\}_{i_0 \leq k \leq m_0-1}$ are square matrices of same length; we deduce that the subvarieties $\{Y_k\}_{i_0 \leq k \leq m_0-1}$ are all isomorphic. $\square$

Let $\alpha \in S - S_\pi$ be a simple root. Denote $P_\alpha$ the minimal standard parabolic subgroup associated to the simple root $\alpha$.

**Definition 3.8.** A projective line of type $\alpha$ is a subset of $G/P$ of the form $g \cdot P_\alpha P/P$, where $g \in G$. 
We can remark that $P_2 P/P \simeq P_2/P_2 \cap P$. But $P_2 \cap P = B$, because $z$ is not in $S_p$. So we get $P_2 P/P \simeq P_2/B \simeq \mathbb{P}^1$. Two projective lines of the same type are disjoint or are equal and two projective lines of different types have at most a common point [26, p. 146].

**Theorem 3.9 (Main theorem).** With the notations of the last theorem. Let $p \geq q$ be two adjacent ordered partitions. Let $x \in O_q \cap n_p$.

(i) If $\text{codim}_{\overline{O}_p}(O_q) = 2$, then $f_p^{-1}(x)$ is a finite union of projective lines: for every $i \in \{ \sum_{u \leq k} \hat{p_u} \}_{i_0 \leq k \leq m_0-1}$ there is a unique projective line of type $z_i$ in $f_p^{-1}(x)$. Moreover, $f_p^{-1}(x)$ is the union of these projective lines which intersect themselves transversely. Finally, the projective lines of type $z_i$ and $z_j$ have a non-empty intersection if and only if $i = \sum_{u \leq k} \hat{p_u}$ and $j = \sum_{u \leq l} \hat{p_u}$ with $l = k \pm 1$. In particular, $f_p^{-1}(x)$ is isomorphic to the Dynkin curve in $A_{m_0-i_0}$.

(ii) If $\text{codim}_{\overline{O}_p}(O_q) > 2$, then $f_p^{-1}(x)$ is reduced to a unique irreducible component isomorphic to the projective space $\mathbb{P}^{\hat{p}_{i_0} - \hat{p}_{m_0} + 1}$.

**Proof.** Like for the proof of the last theorem we will consider two cases.

(i) Case $p_1 > q_1$: for every $\sum_{u \leq i_0} \hat{p_u} \leq k \leq n - 1$, denote $q_k$ the standard parabolic subalgebra whose associated parabolic subgroup $Q_k$ is given by the subset of simples roots $S_p \cup \{ z_k \}$. We notice that $Q_k/P \simeq P_{z_k} P/P$. Moreover, these standard parabolic subalgebras are associated to the dual partition $\hat{q}$ (cf. Proof of the last theorem), by Proposition 2.4 we deduce that $f_p^{-1}(x)$ is a union of projective lines of type $z_k$ for $\sum_{u \leq i_0} \hat{p_u} \leq k \leq n - 1$, and for every type $z_k$ we find a unique projective line of the same type.

Finally, for every $\sum_{u \leq i_0} \hat{p_u} \leq k \leq n - 1$ such that $|k - l| \geq 2$ we can remark that the intersection of the two hyperplanes $\{ p_{k,k+1} = 0 \} \cap n_p$ and $\{ p_{l,l+1} = 0 \} \cap n_p$ in $n_p$ is exactly the nilpotent radical of a standard parabolic corresponding to a partition associated to $\hat{t} = (\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_i, 2, 1, 1, \ldots, 1)$. Then we have $\hat{t} \geq \hat{q}$ and $\hat{t} \neq \hat{q}$. As consequence, we deduce that $O_q \cap n_{q_k} \cap n_q = \emptyset$. By Proposition 2.4 the corresponding projective lines $g_k P_{z_k} P/P$ and $g_l P_{z_l} P/P$ in $f_p^{-1}(x)$ have an empty intersection.

(ii) Case $p_1 = q_1$, by Proposition 3.2 if $\text{codim}_{\overline{O}_p}(O_q) = 2$ we get $\hat{p}_{i_0} = \cdots = \hat{p}_{m_0-1} = \hat{p}_{m_0} \geq 2$, and if $\text{codim}_{\overline{O}_p}(O_q) > 2$ then $m_0 = i_0 + 1$ and we get $\hat{p}_{i_0} > \hat{p}_{i_0+1} \geq 2$.

By Theorem 3.7, the irreducible components $\{ Y_k \}_{i_0 \leq k \leq m_0-1}$ of $n_p - (O_p \cap n_p)$ induce the nilpotent orbit $O_q$. Fix an integer $i_0 \leq k \leq m_0 - 1$, we now compute the irreducible component of $f_p^{-1}(x)$ corresponding to the subvariety $Y_k$ (see Remark 3.6 (ii)).

Denote $b := \hat{p}_k$ and $a := \hat{p}_{k+1}$, we have $b \geq a$ and we get $b = a$ in case $m_0 \geq i_0 + 2$. Denote $i := \sum_{u \leq k} \hat{p_u}$ and let $z_i$ the corresponding simple root. Consider $x = x_1 + x_2 \in Y_k \cap O_q$, with $x_1 = (M^1_1, \ldots, M^l_{i-1}) \in V_p \simeq \text{Mat}_{\hat{p}_1 \times \hat{p}_2}(\mathbb{C}) \times \cdots \times \text{Mat}_{\hat{p}_{i-1} \times \hat{p}_i}(\mathbb{C})$ (cf.
(3.3) and (3.4)) and \( x_2 \in [n_p, n_p] \). By (3.9) we have to choose \( M_k^1 \) of rank \( a - 1 \); for our computation we will choose \( x_1 \) with

\[
M_k^1 = E_{i,i+1} + E_{i-1,i+2} + \cdots + E_{i-a+2,i+a-1},
\]  

which is of rank \( a - 1 \), i.e., \( M_k^1 \) has the following shape:

\[
\begin{bmatrix}
  b \\
  & M_k^1 \\
  a & \end{bmatrix}
\]

\[
M_k^1 = \begin{bmatrix}
  0 & 0 & 0 \\
  0 & 1 & 0 \\
  1 & 0 & \cdots & 0 \\
\end{bmatrix}
\]

Let \( w := w_1 w_2 w_3 \in W \) be the element of the symmetric group defined by

\[
w_1 := (s_{i-a+1}s_{i-a}\cdots s_{i-b+1})(s_{i-a+2}s_{i-a+1}\cdots s_{i-b+2})\cdots (s_{i-1}s_{i-2}\cdots s_{i-(b-a)-1}),
\]

\[
w_2 := s_{i+a-1}s_{i+a-2}\cdots s_i+1,
\]

\[
w_3 := s_{i-(b-a)}s_{i-(b-a)+1}\cdots s_i.
\]

Remark that \( w \) is written with the simple transpositions

\[
s_{i-b+1}, s_{i-b+2}, \ldots, s_{i+a-2}, s_{i+a-1},
\]

then we deduce that

\[
N_b(w) \subset \{ \beta \mid \beta = \alpha_u + \alpha_{u+1} + \cdots + \alpha_v, \ i - b + 1 \leq u \leq v \leq i + a - 1 \}. \quad (3.12)
\]

For every \( i - b + 1 \leq m \leq i - a + 1 \), consider the root \( \beta_m := \alpha_m + \alpha_{m+1} + \cdots + \alpha_{i+a-1} \); then we get:

\[
w^{-1}(\beta_m) = w_3^{-1} w_2^{-1} w_1^{-1}(\beta_m)
\]
Moreover by construction \( w_3 \) is written in a reduced form, then by Springer [25, p. 142] we have

\[
N_b(w_3) = \{ \varepsilon_{m+a-1} + \varepsilon_{m+a} + \cdots + \varepsilon_i \mid i - b + 1 \leq m \leq i - a + 1 \}. \tag{3.13}
\]

We deduce that

\[
w^{-1}(\beta_m) < 0. \tag{3.14}\]

Let \( w \) be the representative of minimal length of the class of \( w \) in \( \mathcal{W}/\mathcal{W}_p \), then we can verify that

\[
N_p(w) = \{ \beta \in \mathcal{R}^+ - \mathcal{R}_p \mid w^{-1}(\beta) < 0 \} \tag{3.15}
\]

by (3.14) we deduce that

\[
\{ \beta_m \mid i - b + 1 \leq m \leq i - a + 1 \} \subset N_p(w). \tag{3.16}
\]

By (3.12) and (3.15) we get

\[
N_p(w) \subset \{ \beta \mid \beta = \varepsilon_u + \varepsilon_{u+1} + \cdots + \varepsilon_v, \; i - b + 1 \leq u \leq i \leq v \leq i + a - 1 \}. \tag{3.17}
\]

On the other hand, by (3.10) we have

\[
M_k^1 = \sum_{k=0}^{a-2} E_{i-k,i+k+1} \in \mathfrak{g}_{\varepsilon_i} \oplus \mathfrak{g}_{\varepsilon_{i-1}+\varepsilon_i+\varepsilon_{i+1}} \oplus \cdots \oplus \mathfrak{g}_{\varepsilon_{i-a+2}+\cdots+\varepsilon_{i+a-1}}. \tag{3.18}
\]

For every \( 0 \leq k \leq a - 2 \) we have

\[
w^{-1}(E_{i-k,i+k+1}) = w_3^{-1}w_2^{-1}w_1^{-1}(E_{i-k,i+k+1}) = w_3^{-1}(E_{i,i+k+1}) = w_3^{-1}(E_{i,i+k+2}) \in w_3^{-1}(\mathfrak{g}_{\varepsilon_{i-1}+\varepsilon_i+\varepsilon_{i+1}}) = \mathfrak{g}_{w_3^{-1}(\varepsilon_{i-1}+\varepsilon_i+\varepsilon_{i+1})}.
\]
By (3.13) we have $w_3^{-1} (x_i + \ldots x_{i+k+1}) > 0$. In particular we deduce that $x \in np \cap n_w \cdot np$, where $n_w$ is a representative of $w$ in $\text{Norm}_G(h)$. Let

$$u := (Id_n + \sum_{m=i-b+1}^{i-a+1} \lambda_m E_{m,i+a-1}) \in \prod_{m=i-b+1}^{i-a+1} U_{x_m + \ldots + x_{i+a}}, \quad (3.19)$$

with $\lambda_m \in \mathbb{C}$. By (3.16) we have $u \in U_{p,w}$. Recall that if $v \in U_{p,w}$ and $Y \in g_{y,i} [26, p. 80]$, with (3.18) and (3.19) we have $u M_k u^{-1} = M_k^1$, and by (3.17) we deduce that

$$uxu^{-1} \in np \cap n_w \cdot np. \quad (3.20)$$

By Theorems 2.1 and 3.7 we have

$$\dim(f_p^{-1}(x)) \leq \frac{1}{2} (\dim(G^x) - \dim(I_p)) = \frac{1}{2} (\dim(G) - \dim(O_q) - \dim(I_p))$$

$$\leq \frac{1}{2} (\dim(O_p) - \dim(O_q)) = \frac{1}{2} (\dim(O_p) - 2\dim(O_q \cap np))$$

$$\leq \dim(np) - \dim(O_q \cap np) = \text{codim}_{np}(Y_k) = b - a + 1.$$  

As $\text{card} (\beta_m | \beta_m = x_m + x_{m+1} + \ldots + x_{i+a-1}, \ i - b + 1 \leq m \leq i + a - 1) = b - a + 1$, and with (3.20) we deduce that $\dim(f_p^{-1}(x)) = b - a + 1$ and the irreducible component of $f_p^{-1}(x)$ corresponding to $Y_k$ is given by the closure in $G/P$ of the subvariety

$$(Id_n + \sum_{m=i-b+1}^{i-a+1} \lambda_m E_{m,i+a-1}) n_w P/P$$

$$= n_{w_1} n_{w_2} (Id_n + \sum_{m=i-b+1}^{i-a+1} \lambda_m E_{m+a-1,i+1}) n_{w_3} P/P.$$ 

Let $r_d$ denote the simple reflection which interchanges $d$ and $d + 1$ of the symmetric group $S_z$, with $z = b - a + 2$ and let $F_{i,j}$ denote the elementary matrix in $\mathfrak{sl}(z, \mathbb{C})$. Consider $P_z$ the maximal parabolic subgroup in $\mathsf{SL}(z, \mathbb{C})$ whose corresponding Weyl subgroup of $S_z$ is generated by $r_1, \ldots, r_{z-2}$. Then we have the following isomorphism:

$$(Id_n + \sum_{m=i-b+1}^{i-a+1} \lambda_m E_{m+a-1,i+1}) n_{w_3} P/P$$

$$\simeq (Id_z + \sum_{m=1}^{z-1} \lambda_m F_{m,z}) n_{r_1 r_2 \ldots r_{z-1}} P_z/P_z.$$
But the right member is exactly the big cell in $\text{SL}(z, \mathbb{C})/P_z$, then we deduce that the irreducible component of $f^{-1}(x)$ corresponding to $Y_k$ is isomorphic to $\text{SL}(z, \mathbb{C})/P_z$ which is exactly the projective space of the hyperplanes in $\mathbb{C}^z$, therefore this irreducible component is isomorphic to $\mathbb{C}^b-a+1$.

If $b = a$ we have $w_1 = s_{i-a+1}s_{i-a+2} \ldots s_i$ and $w_3 = s_i$ and we get

$$
(Id_n + \lambda E_{i-a+1,i+a-1})n_w P/P = n_{w_1}(Id_n + \lambda E_{i,i+a-1})n_{w_2}s_i P/P
$$

$$
= n_{w_1}n_{w_2}(Id_n + \lambda E_{i,i+1})n_{s_i} P/P.
$$

Therefore the corresponding irreducible component is a projective line of type $\mathbb{P}^{b-a+1}$.

Let us show now that if $|k-l| \geq 2$, then $Y_k \cap Y_l \cap O_q = \emptyset$. As $|k-l| \geq 2$ we have $\hat{p}_0 = \hat{p}_{b_0} = \cdots = \hat{p}_{m_0-1} = a$. Let $x = x_1 + x_2 \in Y_k \cap Y_l$, with $x_1 = (M^1_1, \ldots, M^1_{l-1}) \in V_p$ (cf. (3.3) and (3.4)) and $x_2 \in [n_p, n_p]$. In particular, we have rank$(M^1_k) < a$ and rank$(M^1_l) < a$. Let $g$ be an element in $P$. We can write:

$$
g = \begin{pmatrix}
L_1 & \ast & \ldots & \ast \\
0 & L_2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ast \\
0 & \ldots & 0 & L_t
\end{pmatrix},
$$

with $L_i \in \text{GL}(\hat{p}_i, \mathbb{C})$. Because of $|k-l| \geq 2$ and rank$(M^1_k) < a$, rank$(M^1_l) < a$, we can choose $L_k, L_{k-1}, L_t$ and $L_{l-1}$ such that the first column of $L_kM^1_kL_{k-1}$ and $L_lM^1_lL_{l-1}$ in $g \tilde{x} g^{-1}$ is zero, in particular we get $g \tilde{x} g^{-1} \in \mathfrak{p}$ where $\mathfrak{p}$ is the standard parabolic subalgebra corresponding to the partition

$$(\hat{p}_1, \ldots, \hat{p}_{k-1}, \hat{p}_k + 1, \hat{p}_k - 1, \hat{p}_{k-2}, \ldots, \hat{p}_{l-1}, \hat{p}_l + 1, \hat{p}_{l-1} - 1, \hat{p}_{l-2}, \ldots, \hat{p}_t),$$

which is associated to the ordered partition

$$(\hat{p}_1, \ldots, \hat{p}_{i_0-1}, a + 1, a + 1, a, \ldots, a, a - 1, a - 1, \hat{p}_{m_0+1}, \ldots, \hat{p}_t)$$
and the last partition strictly dominates the partition

\[ \hat{q} = (\hat{p}_1, \ldots, \hat{p}_{i_0 - 1}, a + 1, a, a, \ldots, a, \hat{p}_{m_0 + 1}, \ldots, \hat{p}_l). \]

As consequence we find \( Y_k \cap Y_l \cap \mathcal{O}_q = \emptyset \), this means that the irreducible components in \( f_p^{-1}(x) \) corresponding to \( Y_k \) and \( Y_l \) are disjoint.

Let us show now that we have a bijection between the set of the irreducible components \( Y_k \) which induce \( \mathcal{O}_q \) and the set of irreducible components of \( \mathcal{O}_q \cap \mathfrak{n}_p \) (cf. Remark 3.6 (ii)). By Theorem 3.7 we have \( \mathcal{O}_q \cap \mathfrak{n}_p \subset \bigcup_{i_0 \leq k \leq m_0 - 1} Y_k \), as consequence every irreducible component of \( \mathcal{O}_q \cap \mathfrak{n}_p \) is contained in a subvariety \( Y_k \) for a certain integer \( i_0 \leq k \leq m_0 - 1 \), then it suffices to show in each subvariety \( Y_k \) we only have a unique irreducible component of \( \mathcal{O}_q \cap \mathfrak{n}_p \). Case (i) is trivial because properties concerning Richardson orbit. Case (ii): let \( D_i, D_j \) be two irreducible components of \( \mathcal{O}_q \cap \mathfrak{n}_p \) contained in \( Y_k \). Let \( x = x_1 + x_2 \in D_i \) (resp., \( y = y_1 + y_2 \in D_j \)) with \( x_1 \in V_p \) and \( x_2 \in [\mathfrak{n}_p, \mathfrak{n}_p] \) (resp., \( y_1 \in V_p \) and \( y_2 \in [\mathfrak{n}_p, \mathfrak{n}_p] \)). By conjugating \( x \) (resp., \( y \)) by an appropriate element \( g \in P \) (resp., \( g' \in P \)) with a good choice of the Levi component in \( g \) (resp., in \( g' \)) (cf. (3.21)), we can suppose that the writing of \( M_k^l \) in \( x_1 \) (resp., \( y_1 \)) has the configuration (3.11) p. 453. The calculus which followed shows that the irreducible components \( C_i \) and \( C_j \) of \( f_p^{-1}(x) \) corresponding to \( D_i \) and \( D_j \) are isomorphic and as consequence \( D_i \) and \( D_j \) are isomorphic. As \( Y_k \) induces the orbit \( \mathcal{O}_q \), one of the irreducible component of \( \mathcal{O}_q \cap \mathfrak{n}_p \) contained in \( Y_k \) is necessary dense in \( Y_k \), if \( D_i \) is dense in \( Y_k \) we have the same property for \( D_j \), so we necessary have \( D_i = D_j \).

In cases (i) and (ii), if \( |k - l| \geq 2 \) then the irreducible components of \( f_p^{-1}(x) \) associated corresponding to \( Y_k \) and \( Y_l \) have an empty intersection. On the other hand the generalized Springer resolution \( f_p \) is birational and its image \( \text{Im} \ f_p = G.\mathfrak{n}_p \) is a normal variety [3, p. 448], by main Zariski Theorem [12, p. 280], the fibers of \( f_p \) are connected, in particular we deduce that the projective lines in \( f_p^{-1}(x) \) corresponding to \( Y_k \) and \( Y_l \) have a non-empty intersection if and only if \( k = l \pm 1 \), and the proof is complete. \( \square \)

4. Application to the study of a germ’s surface singularity

By keeping the notations of the last Section let \( p \geq q \) be two-adjacent ordered partitions with \( \text{codim}_{\mathcal{O}_p}(\mathcal{O}_q) = 2 \). Let \( x \) be an element in \( \mathcal{O}_q \). This Section consists to rely the description of the singularity \( (\overline{\mathcal{O}_p \cap T_x}, x) \), where \( T_x \) is a transverse slice in \( \mathfrak{g} \) to the orbit \( \mathcal{O}_q \) at the point \( x \) to the study of the fiber \( f_p^{-1}(x) \).

**Definition 4.1.** Let \( M \) be a \( G \)-variety. A transverse slice in \( M \) to the orbit of \( x \) at the point \( x \), is a locally closed subvariety \( T_x \) of \( M \) such that:

(a) \( x \in T_x \);
(b) the morphism \( \varphi : G \times T_x \to M, (g, Y) \mapsto g.Y \) is smooth;
(c) \( T_x \) has the minimal dimension for properties (a) and (b).
As we work with \( C \), then \( \dim(T_x) = \text{codim}_M(G.x) \), moreover if \( M \) is smooth then \( T_x \) is necessary smooth \([20, \text{p.
61}]\).

To give such a transverse slice it is enough to take a vector subspace \( T_x \) which is supplementary to the tangent space of the orbit of \( x \) at the point \( x \).

Let \( x \in O_q \). By Jacobson–Morozov Theorem , there is a semisimple element \( h \) and a nilpotent element \( y \) in \( g \) such that

\[
[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h.
\]

Then by Representation theory of \( \mathfrak{sl}(2, C) \) the subvariety \( T_x := x + g^y \) is supplementary in \( g \) to the orbit of \( O_q \) at the point \( x \), where \( g^y \) is the centralizer of \( y \) in \( g \). Denote:

\[
M := \overline{O_p \cap T_x} \quad \text{and} \quad \hat{M} := f_p^{-1}(S).
\]

**Definition 4.2.**

(a) Let \( M \) be a complex algebraic variety. A desingularization of \( M \) is a morphism \( \pi : \hat{M} \rightarrow M \) such that \( \pi \) is a proper birational morphism and that \( \hat{M} \) is a smooth variety.

(b) The normal variety \( M \) has rational singularities if for every desingularization \( \pi : \hat{M} \rightarrow M \) we have \( R^1\pi_*(O_{\hat{M}}) = \{0\} \).

We have the following result:

**Lemma 4.3.**

(i) \( M \subset O_p \cup O_q \);

(ii) The morphism \( f_p|_{\hat{M}} : \hat{M} \rightarrow M \) is a desingularization of \( M \).

**Proof.** (i) Let us show that the elements in \( M \) come from \( O_p \) and \( O_q \). Let \( g = \bigoplus V_i \) the decomposition of \( g \) as sum of irreducible representations for \( < x, h, y > \mathfrak{sl}(2, C) \). Every \( V_i \) contains a unique vector line \( g_i \) such \( [y, g_i] = 0 \) (cf. \([14, \text{p.
33}]\))

\[ n_i \in \mathbb{Z} \]

the eigenvalue of \( \text{ad}_h \) for the subvariety \( g_i \). By \([14, \text{p.
33}]\) we have \( g^y = \bigoplus g_j \). Denote \( \lambda : \mathbb{C}^* \rightarrow G \) the unique parameter subgroup associated to the semisimple element \( h \) and let \( z = x + v \in x + g^y \). We can suppose that \( g^y \) is the direct sum of certain vector spaces \( g_x \) with \( x \in \mathcal{R}^- \), \([6, \text{p.
45/46}]\), we have \( y \in g_{-2} = \sum_{x(h)=-2} g_{2x} \), then we have \( n_i \leq 0 \). If we write \( v = \sum z_i \) with \( z_i \in g_i \), then we have \( \lambda(t).z = t^2x + \sum t^{n_i}z_i \), where \( z_i \in g_i \), and because of nilpotent orbits are stable under \( \mathbb{C}^* \) we deduce that \( z \in M \), \( t \in \mathbb{C}^* \), \( t^2\lambda(t^{-1}).z = x + \sum t^{2-n_i}z_i \in S \cap O_z \). This shows that \( x \) is in the closure of the orbit of every element of \( M \). But \( x \in O_q \) which is adjacent to \( O_p \), we deduce that \( M \subset O_p \cup O_q \).

(ii) By construction we have locally \( g \simeq T_x \times O_q \), and \( \overline{O_p} \) is locally isomorphic to \( M \times O_q \). As \( \hat{M} = f_p^{-1}(M) \), we have

\[
\hat{M} = \{(Y, g.P) ; \ Y \in S \cap g.n_p \}
\]
and locally $T^*(G/P)$ is isomorphic to the space $\hat{M} \times \mathcal{O}_q$. In particular $\hat{M}$ is smooth. And the map $f_{p|\hat{M}}$ is proper because $G/P$ is complete; so $f_{p|\hat{M}} : \hat{M} \to M$ is a desingularization of $M$. □

Lemma 4.4 (Hinich [13, p. 302]). $(M; x)$ is a normal surface with a rational singularity.

Proof. We can remark that $q^{-1}(\mathcal{O}_p) = G \times M$ and we have two smooth morphisms $\varphi_1 : G \times M \to \mathcal{O}_p$ and $pr_2 : G \times M \to M$ at the point $(1_G, x)$. We deduce that $(\mathcal{O}_p, x)$ is normal if and only if the surface $(M, x)$ is normal [8]. By [3,16] it was shown that every closure of nilpotent orbit in $\mathfrak{sl}(n, \mathbb{C})$ is normal. As consequence $(M, x)$ is a normal surface with an isolated singularity.

By Theorem 5 in [9] we deduce that $G \times M$ has rational singularities.

The following diagram

\[
\begin{array}{ccc}
G \times \hat{M} & \xrightarrow{id \times f_{p|\hat{M}}} & G \times M \\
p_1' & \swarrow \downarrow & \searrow \downarrow p_1 \\
G & & \\
\end{array}
\]

is a simultaneous desingularization of the fibers of $p_1$. By Theorem 3 in [9] we can deduce that $M$ has rational singularities. □

Definition 4.5. A $f : \hat{M} \to (M, x)$ desingularization of a normal surface with a rational singularity $x$ is called minimal if every irreducible component of the exceptional fiber $f^{-1}(x)$ has a self-intersection number different of $-1$.

The minimal desingularization exists up to isomorphism and every desingularization of $(M, x)$ factorizes through the minimal desingularization. Moreover, the normal surfaces with a rational singularity for which every irreducible component of the exceptional fiber has a self-intersection $-2$ are well known and are obtained as quotients of $\mathbb{C}^2$ by finite subgroups of $\mathbf{SL}(2, \mathbb{C})$, [20, p. 72]. Such singularities are called simple or rational double points and are classified by the families $A_r, D_r, E_6, E_7, E_8$.

Here is the main theorem of this last Section.

Theorem 4.6. Let $\mathbf{p} = (p_1, p_2, \ldots, p_l) \geq \mathbf{q} = (q_1, q_2, \ldots, q_k)$ two adjacent ordered partitions such that $\text{codim}_{\mathcal{O}_\mathbf{p}}(\mathcal{O}_\mathbf{q}) = 2$ and $p_1 > q_1$. Let $\mathbf{p}$ be the standard subalgebra corresponding to the partition $\mathbf{p}$. Denote $i_0 := \min\{j ; \hat{p}_j \neq \hat{q}_j\}$ and $m_0 := \min\{j > i_0; \hat{p}_j \neq \hat{q}_j\}$. Let $x \in \mathcal{O}_\mathbf{q}$, and $T_x$ be a transverse slice in $\mathfrak{sl}(n, \mathbb{C})$ to the orbit $\mathcal{O}_\mathbf{q}$ at the point $x$. Let $M := \mathcal{O}_\mathbf{p} \cap T_x$ and $\hat{M} := f_{\mathbf{p}}^{-1}(M)$. Then

(i) The morphism $f_{\mathbf{p}|\hat{M}} : \hat{M} \to M$ is the minimal desingularization of the surface $M$.
(ii) The surface $(M, x)$ is a normal surface with a simple singularity of type $A_{m_0-i_0}$.

The following calculus is exactly the same for which H. Esnault has done in the particular subregular case (see [10; 20, p. 88]). The reasoning is done in a more general context but we can apply only for the case \( p_1 > q_1 \).

By Theorem 3.9, \( f_p^{-1}(x) \) is a finite union of projective lines of type \( z_i \) with \( i \in \{ \sum_{u \leq k} \hat{p}_u \}_{i_0 \leq k \leq m_0 - 1} \). To prove the theorem it remains to compute the self-intersection numbers of these projective lines in \( \hat{M} \), i.e., to compute \( c_1(P_{z_i \circ P/P}, \hat{M}) \) the first Chern class of the normal bundle of each of these projectives lines in \( \hat{M} \). Denote \( N(P_{z_i \circ P/P}/\hat{M}) \) the normal bundle of the projective line \( P_{z_i \circ P/P} \) in \( \hat{M} \). If \( A \subset B \subset C \) are three smooth varieties then we have the short exact sequence of normal bundles:

\[
0 \to N_{A/B} \to N_{A/C} \to N_{B/C} \big|_A \to 0.
\]

We apply the last short exact sequence of normal bundles to the three smooth varieties \( P_{z_i \circ P/P} \subset \hat{M} \subset T^* (G/P) \); but we have seen that \( T^*(G/P) \) is locally trivial (it is locally isomorphic to \( \hat{M} \times \mathcal{O}_q \)), as consequence the restrict normal bundle of \( \hat{M} \) in \( T^*(G/P) \) is isomorphic to the tangent bundle of \( \mathcal{O}_q \), the last one is trivial if we consider a small neighborhood of \( x \) in \( Tx \). As consequence we have to compute \( c_1(P_{z_i \circ P/P}, T^*(G/P)) \).

**Lemma 4.7.** Let \( P \) be standard parabolic subgroup of a semisimple complex algebraic group \( G \). Let \( z_i \in S - S_p \) and denote \( P_i \) the parabolic subgroup corresponding to the subset of simple roots \( S_p \cup \{ z_i \} \). Then the natural map \( T^*(G/P) \cong G \times P_{\mathfrak{n}_p} \to G/P_i, (g.P, Y) \mapsto gP_i \) is a locally trivial \( G \)-fibration and we can identify \( G \times P_{\mathfrak{n}_p} \) with the fiber bundle \( G \times P_{\mathfrak{f}_i} \), where \( F_i := P_{\mathfrak{f}_i} \times P_{\mathfrak{n}_p}, \mathfrak{n}_p \) is the nilpotent radical of \( \text{Lie}(P) \).

**Proof.** The natural morphism

\[
\phi : T^*(G/P) \cong G \times P_{\mathfrak{n}_p} \to G/P_i, (g.P, Y) \mapsto gP_i
\]

is \( G \)-invariant so we have \( G \times P_{\mathfrak{n}_p} \cong G \times P_{\mathfrak{f}_i} \), where \( F_i = \phi^{-1}(e) \) [21, p. 26]. It is easy to verify that

\[
F_i = \{ (g.P, Y) \in G \times P_{\mathfrak{n}_p}; \ Y \in g.\mathfrak{n}_p \text{ and } g.P \subset P_i \}
\]

\[
\cong P_i \times P_{\mathfrak{n}_p}.
\]

We can remark that the projective line \( P_iP/P \cong P_i/P \) is contained in \( F_i \). As consequence we have \( P_i/P \subset F_i \subset T^*(G/P) \) three smooth varieties. We deduce that the normal bundle of \( P_i/P \) in \( T^*(G/P) \) is an extension of the normal bundle of \( P_i/P \) in \( F_i \) and of the restrict normal bundle of \( F_i \) in \( T^*(G/P) \). But

\[
T^*(G/P) \cong G \times P_{\mathfrak{n}_p} \to G/P_i
\]
is a locally trivial fibration over a smooth basis, then the restrict normal bundle of $F_i$ in $T^*(G/P)$ is trivial because it is isomorphic to the trivial bundle with fiber the tangent space to $G/P_i$ at the point $\overline{e}$, so its Chern classes are trivial.

We have to compute $c_1(P_i/P, F_i)$.

**Lemma 4.8.** With the above notations. Let $x \in \mathfrak{n}_P$. Let $P_{xi}P/P$ a projective line in $f_p^{-1}(x)$. Then $c_1(P_{xi}P/P, T^*(G/P)) = -2$.

**Proof.** It remains to compute $c_1(P_{xi}P/P, P_i \times^P \mathfrak{n}_P)$. Let $\mathfrak{n}_{P_i}$ denote the nilpotent radical of Lie($P_i$). Then we have the following $P$-invariant short exact sequence:

$$0 \rightarrow \mathfrak{n}_{P_i} \rightarrow \mathfrak{n}_P \rightarrow \mathfrak{n}_P/\mathfrak{n}_{P_i} \rightarrow 0.$$ 

As consequence we deduce that the short exact sequence of bundles:

$$0 \rightarrow P_i \times^P \mathfrak{n}_{P_i} \rightarrow P_i \times^P \mathfrak{n}_P \rightarrow P_i \times^P (\mathfrak{n}_P/\mathfrak{n}_{P_i}) \rightarrow 0.$$ 

But the $P$-module $\mathfrak{n}_{P_i}$ is obtained as the restriction of the $P_i$-module $\mathfrak{n}_{P_i}$, then the bundle $P_i \times^P \mathfrak{n}_{P_i}$ is trivial. Moreover by hypothesis $\mathfrak{n}_P/\mathfrak{n}_{P_i}$ is vector space of dimension 1 on which $P$ acts via the simple root $\alpha_i$; if we write $P = L_P.U_P$ where $L_P = C.(L_P, L_P)$ is the Levi component of $P$ with $C$ (resp., $(L_P, L_P)$) the center (resp., the derived subgroup) of $L_P$ and $U_P$ is its nilpotent radical. The subgroup $U_P$ and $(L_P, L_P)$ have no characters, the action of $P$ on $\mathfrak{n}_P/\mathfrak{n}_{P_i}$ (induced by the adjoint action) comes from the action of $C$, therefore this action is reduced to the action of the maximal torus whose Lie algebra is $\mathfrak{h}_C$ and the differential of the action of the maximal torus is exactly given by the simple root $\alpha_i$. Then we have

$$P_i \times^P (\mathfrak{n}_P/\mathfrak{n}_{P_i}) \simeq \text{SL}(2, \mathbb{C}) \times^B \mathfrak{n}_2 \simeq T^*(\text{SL}(2, \mathbb{C})/B_2)$$

where $B_2$ is the Borel subgroup of $\text{SL}(2, \mathbb{C})$ and $\mathfrak{n}_2$ is the nilpotent radical of Lie($B_2$). But $\text{SL}(2, \mathbb{C})/B_2 \simeq \mathbb{P}^1$. As consequence we get

$$c_1(P_i/P, P_i \times^P \mathfrak{n}_P) = c_1(T^*(\mathbb{P}^1)) = -2. \quad \square$$

**Remark 4.9.** Our approach is another way to get Theorem 4.6 (ii) which was already obtained in [17].

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References