On intersections of orbital varieties
and components of Springer fiber

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Received 23 September 2003
Available online 24 February 2006
Communicated by Peter Littelmann

Abstract

We consider Springer fibers and orbital varieties for GLn. We show that the irreducible components of an intersection of components of Springer fiber are in bijection with the irreducible components of intersection of orbital varieties; moreover, the corresponding irreducible components in this correspondence have the same codimension. Finally we give a sufficient condition to have an intersection in codimension one.

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Keywords: Flag manifold; Springer fibers; Orbital varieties; Robinson–Schensted correspondence; Schubert cell

1. Introduction

1.1. Let G be a semisimple (connected) complex algebraic group with Lie algebra Lie(G) = g on which G acts by the adjoint action. For g ∈ G and u ∈ g we denote this action by g·u := gug⁻¹.

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¹ Supported in part by the Marie Curie Research Training Network “Liegrits.”
² The author is supported by the Feinberg Graduate School fellowship and the Marie Curie Research Training Network “Liegrits.”

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doi:10.1016/j.jalgebra.2006.01.029
Fix a Cartan subalgebra \( \mathfrak{h} \). Let \( \mathcal{W} \) denote the associated Weyl group. We have the Chevalley–Cartan decomposition of \( \mathfrak{g} \):

\[
\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha,
\]

where \( \mathcal{R} \) is the root system of \( \mathfrak{g} \) relatively to \( \mathfrak{h} \). Let \( \Pi \) be a set of simple roots of \( \mathcal{R} \). Denote \( \mathcal{R}^+ \) (respectively \( \mathcal{R}^- \)) the positive roots (respectively negative roots) (w.r.t. \( \Pi \)). We sometimes prefer the notation \( \alpha > 0 \) (respectively \( \alpha < 0 \)) to designate a positive (respectively negative) root. Let \( \mathfrak{b} := \mathfrak{h} \oplus \sum_{\alpha \in \mathcal{R}^+} \mathfrak{g}_\alpha \) be the standard Borel subalgebra (w.r.t. \( \Pi \)) and \( \mathfrak{n} := \sum_{\alpha \in \mathcal{R}^-} \mathfrak{g}_\alpha \) its nilpotent radical. Let \( B \) be the Borel subgroup of \( G \) with \( \text{Lie}(B) = \mathfrak{b} \).

Let \( G \times^B \mathfrak{n} \) be the space obtained as the quotient of \( G \times \mathfrak{n} \) by the right action of \( B \) given by \( (g, x).b := (gb, b^{-1}.x) \) with \( g \in G, x \in \mathfrak{n} \) and \( b \in B \). By the Killing form we get the following identification \( G \times^B \mathfrak{n} \cong T^*(G/B) \). Let \( g \ast x \) denote the class of \( (g, x) \) and \( \mathcal{F} := G/B \) the flag manifold. The map \( G \times^B \mathfrak{n} \rightarrow \mathcal{F} \times \mathfrak{g}, g \ast x \mapsto (gB, g.x) \) is an embedding which identify \( G \times^B \mathfrak{n} \) with the following closed subvariety of \( \mathcal{F} \times \mathfrak{g} \) (see [16, p. 19]):

\[
\mathcal{Y} := \{ (gB, x) \mid x \in g.\mathfrak{n} \}.
\]

The map \( f : G \times^B \mathfrak{n} \rightarrow \mathfrak{g}, g \ast x \mapsto g.x \) is called the Springer resolution and we have the following commutative diagram:

\[
\begin{array}{ccc}
G \times^B \mathfrak{n} & \xrightarrow{\cong} & \mathcal{Y} \\
\downarrow f & & \downarrow \text{pr}_2 \\
\mathfrak{g} & & \mathfrak{g}
\end{array}
\]

where \( \text{pr}_2 : \mathcal{F} \times \mathfrak{g} \rightarrow \mathfrak{g}, (gB, x) \mapsto x \). The map \( f \) is proper (because \( G/B \) is complete) and its image is exactly \( G.\mathfrak{n} = \mathcal{N} \), the nilpotent variety of \( \mathfrak{g} \) [21].

Let \( x \) be a nilpotent element in \( \mathfrak{n} \). By the diagram above we have:

\[
\mathcal{F}_x := f^{-1}(x) = \{ gB \in \mathcal{F} \mid x \in g.\mathfrak{n} \} = \{ gB \in \mathcal{F} \mid g^{-1}.x \in \mathfrak{n} \}.
\] (1.1)

The variety \( \mathcal{F}_x \) is called the Springer fiber above \( x \) and has been studied by many authors. It was one of the most stimulating subjects during the last three decades, appearing in many areas, for example, in representation theory and singularity theory. But it remains a very mysterious object, and the major difficulty is its geometric description which is known in a few cases. For \( x \) in the regular orbit in \( \mathfrak{g} \) it is reduced to one point. For \( x \) in the subregular orbit in \( \mathfrak{g} \) it is a finite union of projective lines which intersect themselves transversally and is usually called the Dynkin curve, it was obtained by J. Tits (see e.g. [24, Theorem 2, p. 153]). For \( x \) in the minimal orbit its irreducible components are some Schubert varieties [2].

The Springer fibers arise in many contexts. They arise as fibers of Springer’s resolution of singularities of the nilpotent variety in [16,17,21]. In the course of these investigations,

Springer defined $W$-module structures on the rational homology groups $H_\ast(F_x, \mathbb{Q})$ on which also the finite group $A(x) = Z_G(x)/Z_{G(x)}^0$ (where $Z_G(x)$ is a stabilizer of $x$ and $Z_{G(x)}^0$ is its neutral component) acts compatibly. Set $d = \dim(F_x)$, the $A(x)$-fixed subspace $H_{2d}(F_x, \mathbb{Q})^{A(x)}$ of the top homology is known to be irreducible [22].

In [8], D. Kazhdan and G. Lusztig tried to understand Springer’s work connecting nilpotent classes and representations of Weyl groups. Among problems they have posed, the conjecture 6.3 in [8] has stimulated much research into the relation between the Kazhdan–Lusztig basis and the Springer fibers.

1.2. More known for $G = \text{GL}_n$. For $x \in n$ its only characteristic value is 0, so that its Jordan form is completely defined by $\lambda = (\lambda_1, \ldots, \lambda_k)$ a partition of $n$ where $\lambda_i$ is the length of $i$th Jordan block. Arrange the numbers in a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ in the decreasing order (that is $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$) and write $J(x) = \lambda$. In turn an ordered partition can be presented as a Young diagram $D_\lambda$—an array with $k$ rows of boxes starting on the left with the $i$th row containing $\lambda_i$ boxes. In such a way there is a bijection between Springer fibers and Young diagrams.

Fill the boxes of Young diagram $D_\lambda$ with $n$ distinct positive integers. If the entries increase in rows from left to right and in columns from top to bottom we call such an array a Young tableau or simply a tableau of shape $\lambda$. Let $\text{Tab}_\lambda$ be the set of all Young tableaux of shape $\lambda$.

Given $x \in n$ such that $J(x) = \lambda$ by Spaltenstein [18] and Steinberg [26] there is a bijection between components of $F_x$ and $\text{Tab}_\lambda$ (cf. 2.5). For $T \in \text{Tab}_\lambda$ set $F_T$ to be the corresponding component of $F_x$.

For $\text{GL}_n$ the conjecture of Kazhdan and Lusztig mentioned in 1.1 is equivalent to the irreducibility of certain characteristic varieties [1, Conjecture 4]. It was shown to be reducible in general by Kashiwara and Saito [7]. Nevertheless, the description of pairwise intersections of the irreducible components of the Springer fibers is still open. In particular the determination in terms of Young tableaux of a pair of irreducible components with the intersections in codimension 1 is unknown in general. The search of these intersections is the main motivation of our paper. The general answer seems to be beyond our means but we can address these questions in some special cases.

Let us first describe the answers in the special cases which are already known.

1.3. The description of the Springer fiber was completely done for the hook and two-row Young diagrams in [4,27]. P. Lorist studied the Springer fiber of dimension 2, [10]. He showed in that case that all the irreducible components of the Springer fiber are either the product of two projective lines or are ruled surfaces over a projective line with $e = 2$ and he also gave the complete description of the intersection between them; his method is very basic but very cumbersome, it consists of calculations of the different intersections of the Springer fiber with every Schubert cell and then pasting them together.

For one of us this work was motivated by Lorist’s work, by the desire to find a more efficient way of computation of the Springer fiber (cf. [14, p. 108]). The idea is to find the unique Schubert cell which intersects generically with a given irreducible component. Obviously the determination of such Schubert cell depends on the choice of the point above which we are looking at the Springer fiber, another point will generate another Schubert
1.4. Let us return to a semisimple algebraic group $G$. Let $x \in \mathfrak{n}$ be some nilpotent element and let $\mathcal{O}_x = G.x$ be its orbit. Consider $\mathcal{O}_x \cap \mathfrak{n}$. Its irreducible components are called orbital varieties associated to $\mathcal{O}_x$. By Spaltenstein’s construction [19] there is a tight connection between $\mathcal{F}_x$ and $\mathcal{O}_x \cap \mathfrak{n}$. We explain it in 2.1.

In particular, for $G = \text{GL}_n$ the Spaltenstein’s construction provides the bijection between the orbital varieties associated to $\mathcal{O}_x$ and components of $\mathcal{F}_x$. That is let $J(x) = \lambda$ then there is a natural bijection $\phi$ between $\{\mathcal{F}_T\}_{T \in \text{Tab}}$ and the set of orbital varieties associated to $\mathcal{O}_x$. Let us denote the set of orbital varieties by $\{\mathcal{V}_T\}_{T \in \text{Tab}}$, where $\mathcal{V}_T = \phi(\mathcal{F}_T)$. As a straightforward corollary of this construction we get in Proposition 2.2 that the number of irreducible components and their codimensions of $\mathcal{F}_T \cap \mathcal{F}_{T'}$, are equal to the number of irreducible components and their codimensions of $\mathcal{V}_T \cap \mathcal{V}_{T'}$. Thus from our point of view orbital varieties are equivalent to the components of Springer fibre.

1.5. The body of the paper consists of three sections. In Section 2 we explain Spaltenstein’s and Steinberg’s constructions and show that on the level of intersections the components of Springer fibre and orbital varieties are the same objects. Finally in Section 3 we give an sufficient condition to have an intersection in codimension one.
\[ Y_i = \bigcup_{a \in A(x)} Y_{a(\sigma)}, \quad (2.1) \]

in particular \( Y_i \) is equidimensional, \( \dim(Y_i) = \dim(Y_\sigma) \) and one has

**Theorem (Spaltenstein).** \( \mathcal{F}_x \) and \( \mathcal{O}_x \cap \mathfrak{n} \) are equidimensional and

\[
\begin{align*}
\dim(\mathcal{O}_x \cap \mathfrak{n}) + \dim(Z_G(x)) &= \dim(\mathcal{F}_x) + \dim(B), \\
\dim(\mathcal{O}_x \cap \mathfrak{n}) + \dim(\mathcal{F}_x) &= \dim(n), \\
\dim(\mathcal{O}_x \cap n) &= \frac{1}{2} \dim(\mathcal{O}_x).
\end{align*}
\]

2.2. In particular, if \( G = \text{GL}_n \) then \( Z_G(x) \) is connected and \( A(x) \) is trivial so that there exists a bijection \( \pi : \{\mathcal{F}_i\}_{i=1}^k \to \{\mathcal{V}_i\}_{i=1}^k \) where \( \pi(\mathcal{V}_i) := f_1(f_2^{-1}(\mathcal{F}_i)) = \mathcal{V}_i \).

As a straightforward corollary of Spaltenstein’s construction for the case \( \text{GL}_n \) we get

**Proposition.** Let \( x \in \mathfrak{n} \) and let \( \mathcal{F}_1, \mathcal{F}_2 \) be two irreducible components of \( \mathcal{F}_x \) and \( \{\mathcal{E}_i\}_{i=1}^l \) the set of irreducible components of \( \mathcal{F}_1 \cap \mathcal{F}_2 \). Then \( \{\pi(\mathcal{E}_i)\}_{i=1}^l \) is exactly the set of irreducible components of \( \mathcal{V}_1 \cap \mathcal{V}_2 \) and \( \text{codim}_{\mathcal{F}_1}(\pi(\mathcal{E}_i)) = \text{codim}_{\mathcal{V}_1}(\pi(\mathcal{E}_i)) \).

**Proof.** Denote \( \{\mathcal{V}_i\}_{i=1}^s \) the set of irreducible components of \( \mathcal{V}_1 \cap \mathcal{V}_2 \). Put \( \mathcal{Y}_1 \cap \mathcal{Y}_2 := f_2^{-1}(\mathcal{V}_1 \cap \mathcal{V}_2) \). By (2.1) we have \( Y_1 \cap Y_2 = \bigcup_{i=1}^s f_1^{-1}(\mathcal{V}_i) = f_1^{-1}(\mathcal{V}_1) \cap f_1^{-1}(\mathcal{V}_2) = f_2^{-1}(\mathcal{F}_a(1)) \cap f_2^{-1}(\mathcal{F}_a(2)) \), since \( A(x) \) is trivial we have \( Y_1 \cap Y_2 = f_2^{-1}(\mathcal{F}(1)) \cap f_2^{-1}(\mathcal{F}(2)) = \bigcup_{i=1}^l f_2^{-1}(\mathcal{E}_i) \), where \( \{\mathcal{E}_i\}_{i=1}^l \) is the set of irreducible components of \( \mathcal{F}_1 \cap \mathcal{F}_2 \). In the same spirit as before each subset \( f_2^{-1}(\mathcal{E}_i) = \theta^{-1}(\mathcal{E}_i) \) is irreducible and we have

\[
\dim(f_2^{-1}(\mathcal{E}_i)) = \dim(\mathcal{E}_i) + \dim(B) \quad (2.2)
\]

and for \( i = 1, 2 \)

\[
\dim(f_2^{-1}(\mathcal{F}_i)) = \dim(\mathcal{F}_i) + \dim(B). \quad (2.3)
\]

If \( f_2^{-1}(\mathcal{E}_i) \subset C \), where \( C \) is an irreducible component of \( Y_1 \cap Y_2 \), then \( \theta(C) \) is irreducible and we necessary have \( \theta(f_2^{-1}(\mathcal{E}_i)) = \theta(\theta^{-1}(\mathcal{E}_i)) = \mathcal{E}_i \subset \theta(C) \), therefore we have \( \mathcal{E}_1 = \mathcal{E}_2 = \theta(C) \), \( C = f_2^{-1}(\mathcal{E}_i) \) and \( \{f_2^{-1}(\mathcal{E}_i)\}_{i=1}^l \) is exactly the set of distinct irreducible components of \( Y_1 \cap Y_2 \). We can suppose that \( x \in \mathcal{V}_1 \cap \mathcal{V}_2 \), then if we notice that \( f_1 \) is the restriction of the orbit map \( \varphi : G \to \mathcal{O}_x, g \mapsto g^{-1}xg \) which is open, we deduce that \( f_1(f_2^{-1}(\mathcal{E}_i)) \) is closed and irreducible in \( \mathcal{V}_1 \cap \mathcal{V}_2 \). We can also easily deduce that \( \{f_1(f_2^{-1}(\mathcal{E}_i))\}_{i=1}^l \) is the set (maybe redundant) of irreducible components of \( \mathcal{V}_1 \cap \mathcal{V}_2 \), therefore \( t \leq s \).

On the other hand, the identity \( Z_G^0(x)f_2^{-1}(\mathcal{E}_i)B = f_2^{-1}(\mathcal{E}_i) \) gives us a natural action of \( \tilde{A}(x) := Z_A(\mathcal{F}_1) \cap Z_A(\mathcal{F}_2) \) on the set \( \{f_2^{-1}(\mathcal{E}_i)\}_{i=1}^l \). Moreover, for any \( g \in G_x \) we have

\[ f_1^{-1}(f_1(g)) = \varphi^{-1}(\varphi(g)) = Z_G(x)g, \]

therefore we have \( f_1^{-1}(f_1(f_2^{-1}(\mathcal{E}_i))) \cap Y_1 \cap \mathcal{E}_2 = \emptyset \).
Let \( g \) be our \( n \).

For \( \alpha_i \in \Pi \), let \( P_{\alpha_i} \) be the standard parabolic subgroup of \( \text{GL}_n \) with \( \text{Lie}(P_{\alpha_i}) = b \oplus g_{-\alpha_i} = b \oplus g_{i+1,i} \). Let \( M_{\alpha_i} \) be the unipotent radical of \( P_{\alpha_i} \) and \( m_{\alpha_i} := \text{Lie}(M_{\alpha_i}) = \bigoplus_{1 \leq s < t \leq n, \ (s,t) \neq (i,i+1)} \mathfrak{g}_{s,t} \).

2.4. Let us return to the parametrization of the components of \( \mathcal{F}_x \) in \( \text{GL}_n \) by standard Young tableaux. But first a few general remarks.

The group \( G \) operates diagonally on \( \mathcal{F} \times \mathcal{F} \) and one version of the Bruhat’s lemma says that the \( G \)-orbits are parameterized by the elements of the Weyl group \( \mathcal{W} \) [23, p. 146]. More precisely, putting

\[
O(w) := \{(gB, g'B) \in \mathcal{F} \times \mathcal{F} \mid g^{-1}g' \in BwB\},
\]

This simple proposition shows that in \( G = \text{GL}_n \) orbital varieties associated to \( O_v \) are equivalent to the components of \( \mathcal{F}_x \).
we have a decomposition into $G$-orbits

$$\mathcal{F} \times \mathcal{F} = \bigsqcup_{w \in \mathcal{W}} O(w).$$

If $Y$ and $Z$ are two irreducible subvarieties of $\mathcal{F}$, then there is a unique $O(w)$ such that $O(w) \cap Y \times Z$ is an open dense set of $Y \times Z$, and we say that $Y$ and $Z$ are in relative position with respect to $w$.

In $\text{GL}_n$, the relative position can be interpreted as follows. If irreducible subvarieties $Y$ and $Z$ of $\mathcal{F}$ are in relative position with respect to $w$ then for two generic flags $\mathcal{F}_1 = (V_1, \ldots, V_n) \in Y$ and $\mathcal{F}_2 = (V'_1, \ldots, V'_n) \in Z$ there exists a basis $\{v_i\}_{i=1}^n$ of $\mathbb{C}^n$ such that for any $j$: $1 \leq j \leq n$ one has $\{v_i\}_{i=1}^j$ is a basis of $V_j$ and $\{v_{w(i)}\}_{i=1}^j$ is a basis of $V'_j$.

2.5. Now we restrict to $g = \mathfrak{sl}_n$, then $\mathcal{N}$ is the variety of all nilpotent matrices, $\mathcal{F}$ is identified with the set of complete flags $\xi = (V_1 \subset \cdots \subset V_n = \mathbb{C}^n)$ and $\mathcal{F}_x \cong \{\xi = (V_i) \in \mathcal{F} \mid x(V_i) \subset V_i-1\}$.

Recall notation from 1.2. Given $x \in n$ let $J(x) = \lambda$. By a slight abuse of notation we will not distinguish between the partition $\lambda$ and its Young diagram. By R. Steinberg [26] and N. Spaltenstein [18] we have a parametrization of the irreducible components of $\mathcal{F}_x$ by the set $\text{Tab}_\lambda$: Let $\xi = (V_i) \in \mathcal{F}_x$, then we get a sutured chain

$$\text{St}(\xi) := (Y(x), Y(x|_{V_{n-1}}), \ldots, Y(x|_{V_1}), Y(x|_{V_1}))$$

in the poset of Young diagrams (where $x|_{V_i}$ is the nilpotent endomorphism induced by $x$ by restriction to the subspace $V_i$). Note that $J(x|_{V_{i+1}})$ differs from $J(x|_{V_i})$ by one corner box, put $i + 1$ in it. It is easy to see that in such a way we get a standard Young tableau corresponding to the given chain. So we get a map $\text{St} : \mathcal{F}_x \rightarrow \text{Tab}_\lambda$. Then the collection $\{\text{St}^{-1}(T)\}_{T \in \text{Tab}_\lambda}$ is a partition of $\mathcal{F}_x$ into smooth irreducible subvarieties of the same dimension and $\{\text{St}^{-1}(T)\}_{T \in \text{Tab}_\lambda}$ is the set of the irreducible components of $\mathcal{F}_x$. Let us denote $\mathcal{F}_\lambda := \mathcal{F}_x$ if $J(x) = \lambda$ and the components of $\mathcal{F}_\lambda$ by $\mathcal{F}_T := \text{St}^{-1}(T)$ where $T \in \text{Tab}_\lambda$.

On the level of orbital varieties the construction is as follows. Consider the canonical projections $\pi_{1,n-i} : n_n \rightarrow n_{n-i}$ acting on a matrix by deleting the last $i$ columns and the last $i$ rows. Given $x \in n$ with $J(x) = \lambda$ for any $u \in \mathcal{O}_x \cap n$ set $J_n(u) := J(u) = \lambda$ and $J_{n-i}(u) := J(\pi_{1,n-i}(u))$ for any $i$: $1 \leq i \leq n - 1$. Exactly as in the previous construction we get a standard Young tableau corresponding to the chain $(J_n(u), \ldots, J_1(u))$, so that $\text{St}_1 : \mathcal{O}_x \cap n \rightarrow \text{Tab}_\lambda$. Again the collection $\{\text{St}_1^{-1}(T)\}_{T \in \text{Tab}_\lambda}$ is a partition of $\mathcal{O}_x \cap n$ into smooth irreducible subvarieties of the same dimensions and $\{\text{St}_1^{-1}(T) \cap \mathcal{O}_x\}_{T \in \text{Tab}_\lambda}$ are the set of the irreducible components of $\mathcal{O}_x \cap n$. Let us denote $\mathcal{O}_\lambda := \mathcal{O}_x$ if $J(x) = \lambda$ and orbital varieties associated to $\mathcal{O}_\lambda$ by $\mathcal{V}_T := \text{St}_1^{-1}(T) \cap \mathcal{O}_\lambda$ where $T \in \text{Tab}_\lambda$.

2.6. A general construction for orbital varieties by R. Steinberg (cf. [25]) is as follows. For $\alpha \in R$ let $\mathfrak{g}_\alpha$ denote the root space.
For $w \in \mathcal{W}$ consider the subspace

$$n \cap^w n := \bigoplus_{\alpha \in R_+ \cap^w R_+} g_{\alpha}$$

of $n$. Then $G.(n \cap^w n)$ is an irreducible locally closed subvariety of $N$. Since the nilpotent variety is a finite union of nilpotent orbits, it follows that there is a unique nilpotent orbit $O_w$ such that $G.(n \cap^w n) = \overline{O_w}$. By [25] $\mathcal{V}_w := \overline{B.(n \cap^w n)} \cap \mathcal{O}_w$ is an orbital variety associated to $\mathcal{O}_w$ and the map $\varphi : w \mapsto \mathcal{V}_w$ is a surjection of $W$ onto the set of all orbital varieties. According to the map $\varphi$, we decompose the Weyl group into the subsets $C_w := \{v \in W \mid \mathcal{V}_v = \mathcal{V}_w\}$ which are called the geometric cells of $W$.

Let $P_{\mathcal{V}_w}$ be the maximal standard parabolic subgroup of $G$ stabilizing $\mathcal{V}_w$. Set $\tau(\mathcal{V}_w) := \{\alpha \in \Pi : P_{\alpha} \mathcal{V}_w = \mathcal{V}_w\}$. Obviously, $P_{\mathcal{V}_w} = \langle P_{\alpha} : \alpha \in \tau(\mathcal{V}_w) \rangle$. Set $\tau(w) := \{\alpha \in \Pi : w^{-1}(\alpha) \in R^-\}$. By [5, §9] one has $\tau(\mathcal{V}_w) = \tau(w)$. In particular, $\tau(w) = \tau(y)$ for any $y \in C_w$ and we can define $\tau(C_w) := \tau(w)$.

Denote $R(w) := \{\alpha \in R^+: w^{-1}(\alpha) < 0\}$ and $S(w) := \{\alpha \in R^+: w^{-1}(\alpha) > 0\}$. Here is a very useful lemma

**Lemma.** Fix a simple root $\alpha$. Denote $l(\ )$ the length function:

1. If $l(s_{\alpha}w) = l(w) + 1$, then $S(s_{\alpha}w) = s_{\alpha}(S(w)) - \{\alpha\}$.
2. If $l(s_{\alpha}w) = l(w) - 1$, then $S(s_{\alpha}w) = s_{\alpha}(S(w)) \cup \{\alpha\}$.
3. If $l(ws_{\alpha}) = l(w) + 1$, then $S(ws_{\alpha}) = S(w) - \{w(\alpha)\}$.
4. If $l(ws_{\alpha}) = l(w) - 1$, then $S(ws_{\alpha}) = S(w) \cup \{w(-\alpha)\}$.

**Proof.** If $l(s_{\alpha}w) = l(w) + 1$ and if $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression for $w$ then $s_{\alpha}s_{i_1} \cdots s_{i_k}$ is also a reduced expression for $s_{\alpha}w$, then by [23, p. 142] we have

$$R(w) = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \ldots, s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})\}$$

and

$$R(s_{\alpha}w) = \{\alpha, s_{\alpha}(\alpha_{i_1}), s_{\alpha}s_{i_1}(\alpha_{i_2}), \ldots, s_{\alpha}s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})\}.$$  \hspace{1cm} (2.7)

Therefore we get $R(s_{\alpha}w) = \{\alpha\} \cup s_{\alpha}(R(w))$; on the other hand, we have

$$R^+ = R(s_{\alpha}w) \bigsqcup S(s_{\alpha}w) = (\{\alpha\} \cup s_{\alpha}(R(w))) \bigsqcup S(s_{\alpha}w) = R(w) \bigsqcup S(w).$$  \hspace{1cm} (2.9)

moreover, we have

$$s_{\alpha}(R^+) = (R^+ - \{\alpha\}) \cup \{-\alpha\} = s_{\alpha}(R(w)) \bigsqcup s_{\alpha}(S(w)).$$  \hspace{1cm} (2.10)

By (2.9) and (2.10) we deduce that $S(s_{\alpha}w) = s_{\alpha}(S(w)) - \{\alpha\}$. The other cases can be obtained in the same manner. \hspace{1cm} $\square$
Let us consider Steinberg’s construction in $\mathfrak{sl}_n$. Here $W = S_n$ where we identify $s_{\alpha_i} := (i, i + 1)$ (in the cyclic form). We write an element $w \in S_n$ in a word form $w = [a_1, \ldots, a_n]$ where $w(i) = a_i$. In what follows we denote $s_i := s_{\alpha_i}$.

Put $p_w(i) := w^{-1}(i)$ to be its position in the word $w$. By [6, 2.3] one has

**Proposition.** For any $w \in S_n$

$$n \cap^w n = \bigoplus_{1 \leq i < j \leq n \atop p_w(i) < p_w(j)} g_{i,j}.$$

In particular, $\tau(w) = \{\alpha_i : p_w(i) > p_w(i + 1)\}$.

Let us describe the geometric cells in the case $G = \text{GL}_n$. In The Robinson–Schensted correspondence gives the bijection from the ordered pairs of standard Young tableaux of the same shape onto the $S_n$ (cf. [3], for example). Let us denote it by $\text{RS} : \bigsqcup_{\lambda \vdash n} \text{Tab}_{\lambda} \times \text{Tab}_{\lambda} \to S_n$ and describe it in short. Let $(T, T')$ be the pair of Standard Young tableaux of the same shape. Remove the number $n$ (and the cell that contains it) from $T'$. Then take the number which is in the same position in $T$ as $n$ was in $T'$ and move it up one row to displace the largest number in that row that is smaller than it; use the displaced number to displace a number in the next higher row according to the same rule, and so on, until a number $r_n$, is displaced from the first row; set $\text{RS}(T, T')(n) = r_n$. Note that the two new tableaux of size $n - 1$ are again of the same shape and the second tableau is standard. Repeat the process to get $\text{RS}(T, T')(n - 1) = r_{n-1}$ and so on. Repeating this procedure $n$ times we get the required element $\text{RS}(T, T')$. We will write it in a word form $\text{RS}(T, T') = [r_1, \ldots, r_n]$.

$S_n$ is decomposed into Young cells where a Young cell corresponding to $T \in \text{Tab}_{\lambda}$ is defined by $C_T := \{\text{RS}(T, T') : T' \in \text{Tab}_{\lambda}\}$. By [25, §5] one has (cf. [11, p. 201], for example).

**Theorem.** Let $w = \text{RS}(T, T')$ where $T, T'$ are of shape $\lambda$. Then

1. $\mathcal{O}_w = \mathcal{O}_{\lambda}$;
2. $\mathcal{V}_w = \mathcal{V}_T$;
3. $\mathcal{C}_w = C_T$.

Note also that the two constructions we gave in $\text{GL}_n$ coincide, namely (cf., for example, [12, 3.4]). Moreover, we can notice that the geometric cells coincide with the Young cells.

**Proposition.** Let $x \in n \cap^w n$ and $T \in \text{Tab}_{\lambda}$. Then for any $w = \text{RS}(T, T')$ one has $B.(n \cap^w n) \cap \text{St}^{-1}_1(T)$ is dense in $B.(n \cap^w n)$.

Let us mention a few well-known combinatorial facts concerning Robinson–Schensted procedure.

2.10. Let us consider Steinberg’s construction in $\mathfrak{sl}_n$. Here $W = S_n$ where we identify $s_{\alpha_i} := (i, i + 1)$ (in the cyclic form). We write an element $w \in S_n$ in a word form $w = [a_1, \ldots, a_n]$ where $w(i) = a_i$. In what follows we denote $s_i := s_{\alpha_i}$.

Put $p_w(i) := w^{-1}(i)$ to be its position in the word $w$. By [6, 2.3] one has

**Proposition.** For any $w \in S_n$

$$n \cap^w n = \bigoplus_{1 \leq i < j \leq n \atop p_w(i) < p_w(j)} g_{i,j}.$$

In particular, $\tau(w) = \{\alpha_i : p_w(i) > p_w(i + 1)\}$.
Let $C'_T := \{RS(T', T) : T' \in \text{Tab}_\lambda\}$. Let $C_\lambda := \{w \in S_n : O_w = O_\lambda\}$ Obviously, $C_\lambda = \bigsqcup_{T \in \text{Tab}_\lambda} C_T = \bigsqcup_{T' \in \text{Tab}_\lambda} C'_T$.

Given $T' \in \text{Tab}_\lambda$ put $r_T(j)$ to be the number of the row $j$ belongs to and $c_T(j)$ to be the number of the column $j$ belongs to.

**Proposition.**

(1) $\tau(C_T) = \{\alpha_i : r_T(i) < r_T(i + 1)\}$.
(2) $(C'_T) = \{w^{-1} : w \in C_T\}$.
(3) Let $w = RS(T, T')$ and let $\lambda$ be the shape of $T$. If $wsi \in C_\lambda$ (respectively $siw \in C_\lambda$) for some $i$ then $wsi \in C_T$ (respectively $siw \in C'_T$).

**Proof.** We give a short proof for the completeness.

(1) The first result is a straightforward corollary of RS algorithm and of Proposition 2.7.
(2) The second result is a straightforward corollary of the Robinson–Schensted theorem (cf. [9, 5.1.4], for example) claiming $(RS(T, T'))^{-1} = RS(T', T)$.
(3) If $l(ws_i) = l(w) + 1$, by Lemma 2.6(3), one has $n \cap ws_i n \subset n \cap w n$ so that $\overline{V}_{ws_i} \subset \overline{V}_w$. On the other hand, by equidimensionality of orbital varieties associated to $O_\lambda$ one has $\dim V_w = \dim V_{ws_i}$. Thus $V_w = V_{ws_i}$, i.e. $w, ws_i \in C_T$. Now if $l(ws_i) = l(w) - 1$ then $w = ys_i$ where $y = ws_i$ and $l(w) = l(y) + 1$ so that by the previous $V_w = V_y$.

The result for $w, s_iw$ is obtained by applying (2). □

For a tableau $T$ we put $\tau(T) := \{\alpha_i : r_T(i) < r_T(i + 1)\}$. By the proposition above one has $\tau(T) = \tau(C_T)$.

2.11. In [26] R. Steinberg gives also a very beautiful interpretation of the relative position between the irreducible components of $F_\lambda$ by the Robinson–Schensted correspondence. Let $T, T' \in \text{Tab}_\lambda$ and let $F_T, F_{T'}$ be the corresponding components of $F_\lambda$. Then by [26] the relative position between the irreducible components $F_T$ and $F_{T'}$ is exactly $RS(T, T')$.

2.12. Recall the Bruhat–Tits decomposition of the flag manifold:

$$F = \bigsqcup_{w \in S_n} X_w.$$  

Where $X_w := B.(w(\xi_0))$ is the $B$-orbit of the flag $w(\xi_0)$ where $\xi_0$ is the canonical flag. It is well known that $X_w$ is an affine space called the Schubert cell (associated to $w$) and its closure $\overline{X}_w$ is called a Schubert variety (cf. [23, p. 149]).

Let $C$ be an irreducible subvariety of $F$, then there is a unique Schubert cell $X_w$ such that $X_w \cap C$ in an open dense subset of $C$. We will call the element $w$ the *position* of $C$ in the flag manifold $F$ (w.r.t. $(\mathfrak{h}, b)$).

2.13. Note also the following straight connection between Steinberg’s construction and relative position:
Theorem. Let $T \in \text{Tab}_\lambda$ and let $w = \text{RS}(T, T')$. Then for a general element $x \in V_T \cap B,(n \cap^w n)$ the position of the irreducible component $F_{T'}$ of the Springer fiber $F_x$ is given by $w$.

Proof. Let $x \in V_T \cap B,(n \cap^w n)$ be in a general position. Let $F_{T'}$ be an irreducible component of the Springer fiber $F_x$ above $x$, and denote $w$ its position. Then $X_w \cap F_{T'}$ is an open dense subset of $F_{T'}$ and by the Bruhat–Tits decomposition any element $\xi = gB \in X_w \cap F_{T'}$ can be written as $g = bn_w b'$ where $n_w$ is a representative of $w$ in $\text{Norm}_G(\mathfrak{h})$ and we can assume that $b' = e$. By (1.1) we have

$$gB \in F_x \iff x \in g.n$$
$$\iff x \in bn_w.n$$
$$\iff b^{-1}xb \in n \cap^w n$$
$$\iff x \in B,(n \cap^w n).$$

Note that by [25, Corollary 3.9.] $x \in V_T \cap B,(n \cap^w n)$ being in a general position is equivalent to choose $gB$ in a general position in $F_{T'}$.

Because of the fact that $x$ is in a general position in $V_T$ we may assume $x \in n(T)$ by 2.9, so we get $\xi_0 \in F_T$. Now the key point is to observe that we can choose $x$ generically in $n(T)$ such that $\xi_0$ is also in general position in $F_T$, and the proof is complete. \qed

Remarks.

(1) Thus, the Young cell corresponding to $T$ describes generically the different positions of the irreducible components of the Springer fiber above the orbital variety $V_T$.

(2) The last theorem is a natural generalization of a result obtained in [15]: Let $B = (e_1, \ldots, e_n)$ a base of $\mathbb{C}^n$ such that $E_i = (e_1, \ldots, e_i)$. A nilpotent element $x$ is said to be adapted to $B$ if the matrix of $x$ in $B$ is a Jordan matrix with decreasing block sizes. Let $T_{\text{max}}$ denote the standard tableau obtained by filling first line of the Young diagram $Y(\lambda)$ with the integers $\{1, \ldots, \lambda_1\}$, the second one with the integers $\{\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2\}$, and so on. Then we have $\xi_0 \in F_{T_{\text{max}}}$, moreover, we have shown that the irreducible component $F_{T_{\text{max}}}$ contain a dense orbit under the centralizer of $x$, this property is not true in general (cf. [25, Remark 5.7. (d)]). As it was explained in [14], the choice of $x$ in the Jordan form is done to have a computation of the Springer fiber easier.

3. Some intersections of codimension one

3.1. In this section we start to consider the orbital varieties (respectively components of Springer fiber) of codimension 1.

For this last section we give a very simple sufficient condition for two orbital varieties associated to $O_\lambda$ (respectively two components of $F_x$) to intersect in codimension 1.
Proposition.

(1) If \( \alpha_k \in \tau(w) \) then \( B.\mathbf{s}_k (n \cap w \, n) \leq V_w, \quad B.\mathbf{s}_k (n \cap w \, n) \cap O_w \leq V_w \) and \( \text{codim}_{V_w} B.\mathbf{s}_k (n \cap w \, n) \cap O_w \leq 1. \)

(2) If \( \alpha_k \notin \tau(w) \) and \( O_w = O_{skw} \) then \( \text{codim}_{V_w} V_w \cap V_{skw} = 1. \)

Proof. (1) If \( \alpha_k \in \tau(w) \), denote \( n_{sk} \) a representative of \( s_k \) in \( \text{Norm}_G(h) \); we have by 2.6 that \( V_w \) is \( P_{\alpha_k} \)-stable, and since \( n_{sk} \in P_{\alpha_k} \), we have \( \mathbf{s}_k (n \cap w \, n) = n_{sk} (n \cap w \, n) \leq P_{\alpha_k} \cdot (n \cap w \, n) = B.(n \cap w \, n) = V_w, \) so we deduce that \( B.\mathbf{s}_k (n \cap w \, n) \leq V_w. \) On the other hand, we have \( B.\mathbf{s}_k (n \cap w \, n) \leq P_{\alpha_k} \cdot \mathbf{s}_k (n \cap w \, n) = P_{\alpha_k} \cdot (n \cap w \, n) = V_w, \) and since \( \text{codim}_{P_{\alpha_k}} B = 1 \) we get

\[
\text{codim}_{P_w} B.\mathbf{s}_k (n \cap w \, n) \leq 1. \quad (3.1)
\]

Since \( V_w = B.(n \cap w \, n) \cap O_w, \) we deduce in particular that \( (n \cap w \, n) \cap O_w \neq \emptyset \) and \( (n \cap w \, n) \subseteq \overline{O_w}, \) so \( s_k (n \cap w \, n) \cap O_w \neq \emptyset \) and \( s_k (n \cap w \, n) \subseteq \overline{O_w}. \) The subvariety \( B.\mathbf{s}_k (n \cap w \, n) \) is irreducible and is contained in the nilpotent variety, there is a unique nilpotent orbit \( O \) such that \( B.\mathbf{s}_k (n \cap w \, n) \cap O \) is open and dense in \( B.\mathbf{s}_k (n \cap w \, n), \) and by the analysis did before we necessary have \( O = O_w \) and \( B.\mathbf{s}_k (n \cap w \, n) \cap O_w \leq V_w, \) and with (3.1) we get

\[
\text{codim}_{P_w} B.\mathbf{s}_k (n \cap w \, n) = \text{codim}_{P_w} B.\mathbf{s}_k (n \cap w \, n) \cap O_w \leq 1. \quad (3.2)
\]

(2) If \( \alpha_k \notin \tau(w) \) (i.e. \( l(w) = l(s_k w) - 1 \)) then by Lemma 2.6(2) we get \( n \cap w \, n = \mathbf{g}_{k,k+1} \oplus s_k (n \cap s_k w \, n) \), then \( B.\mathbf{s}_k (n \cap s_k w \, n) \leq B.(n \cap w \, n) = V_w, \) and as before we have also \( B.\mathbf{s}_k (n \cap s_k w \, n) \cap O_w \subseteq V_w. \) If \( O_{skw} = O_w \) then with the analysis did in (1) for the case \( \alpha_k \in \tau(s_k w) \) we have \( B.\mathbf{s}_k (n \cap s_k w \, n) \cap O_w \subseteq V_{skw}, \) therefore \( B.\mathbf{s}_k (n \cap w \, n) \cap O_w \subseteq V_w \cap V_{skw} \) and since \( V_w \neq V_{skw} \) with (3.2) we get

\[
\text{codim}_{V_{skw}} B.\mathbf{s}_k (n \cap w \, n) \cap O_{skw} = \text{codim}_{V_w} B.\mathbf{s}_k (n \cap w \, n) \cap O_w = 1. \quad (3.3)
\]

Actually we can also deduce the last result from the work of J. Tits: Let \( x \in n \) a nilpotent element. Consider an element \( \xi = gB \in F_x, \) by the Bruhat–Tits decomposition we write \( g = bn_w b' \) and we can assume that \( b' = e. \) Write \( w = s_1 \cdots s_k, \) where \( s_j \) is the reflexion with respect to the simple root \( \alpha_j \in S \) and \( k \) is minimal (i.e. \( w = s_1 \cdots s_k \) is a reduced expression for \( w, \) in particular we have \( w(\alpha_k) < 0 \)). Denote \( g_1 = bn_w w' \) where \( w' = s_1 \cdots s_{k-1}, \) and \( P_k \) the minimal parabolic subgroup containing \( B \) associated to the simple root \( \alpha_k. \) Then the projective line \( g_1 P_k B \) in \( F \) joins the two points \( gB \) and \( g_1 B, \) moreover, J. Tits showed that \( g_1 P_k B \) lies in \( F_x \) (cf. [21, p. 377] or [24, Proposition 1, p. 131]). In particular, if \( w \) corresponds to the position of the irreducible component \( F_T, \) then \( F_T \) is a union of projective lines of type \( \alpha_k, \) i.e. the natural projection \( \pi_k : G/B \to G/P_k \) induces a structure of \( \mathbb{P}^1 \)-bundle on \( F_T \) (see e.g. [20, Lemme 1.11.]).

Consider the morphism

\[
\pi_w : X_w \cap F_T \to F_x, \quad gB \mapsto g_1 B \quad (3.4)
\]
which consists to “flat” the irreducible component $F_T$ under the direction $\alpha_k$, then $\text{Im}(\pi_w)$ is an irreducible subvariety of codimension 1 in $F_T$. In particular if $w'$ is the position of an other irreducible component $F_{T'}$, then $F_T$ and $F_{T'}$ have an intersection of codimension 1.

If $O_w = O_{skw}$, then by Proposition 2.10(3) there exist $T, T', T'' \in \text{Tab}_\lambda$ such that $w = \text{RS}(T, T'')$ and $skw = \text{RS}(T', T'')$. By the last proposition we have

$$\text{codim}_{V_T} (V_T \cap V_{T'}) = \text{codim}_{V_T} (V_T \cap V_{T'}) = 1.$$  

By Proposition 2.2, we also have

$$\text{codim}_{F_T} (F_T \cap F_{T'}) = \text{codim}_{F_T} (F_T \cap F_{T'}) = 1.$$  

This is coherent with the description we did just above with the work of J. Tits: Indeed by Theorem 2.13, $w^{-1}$ and $w^{-1} sk$ are exactly the positions of the irreducible components $F_T$ and $F_{T'}$ above the orbital variety $V_{T''}$.

Remarks.

(1) Thus, if there exists $T'' \in \text{Tab}_\lambda$ such that $\text{RS}(T'', T') = \text{RS}(T'', T)s_k$ for some $s_k$, then $F_T$ and $F_{T'}$ have an intersection in codimension one.

(2) The computation in low rank cases and the full picture in hook case described in [27] gives an impression that $\text{codim}_{F_T} (F_T \cap F_{T'}) = 1$ if and only if there exists $T'' \in \text{Tab}_\lambda$ such that $\text{RS}(T', T'') = s_k \text{RS}(T, T'')$ for some $s_k$. However this is not true in general as we show in [13]. The problem of defining all possible pairs $T, T' \in \text{Tab}_\lambda$ such that $\text{codim}_{F_T} (F_T \cap F_{T'}) = 1$ in terms of Young tableaux only is very tricky.

Acknowledgments

We thank V. Hinich for some very important geometric discussions and A. Joseph for his explanations on the subject. The second author express his deep gratitude to A. Joseph for the invitation to the Weizmann Institute of Science. He also thanks the Weizmann Institute of Science for its support and hospitality.

References